CLASSIFYING DOUBLY-INVARIANT SUBSPACES

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Thesis

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## ABSTRACT

Fix a prime $p$ and consider the vector space of dimension $p^2$ over the field of $p$ elements. We shall define two nilpotent linear transformations on this vector space. We are interested in enumerating and computing the subspaces which are simultaneously invariant under both transformations. We shall do this completely for the cases $p = 2$ and $p = 3$. The case $p = 5$ is computationally much larger and we have only partially completed it in this thesis.

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# CHAPTER I

# INTRODUCTION

For the purposes of this thesis, we shall fix a prime $p$ and consider the vector space of dimension $p^2$ over the field of $p$ elements. Further, we shall define two nilpotent linear transformations on this vector space. The main point of interest of this thesis is enumerating and computing the subspaces which are simultaneously invariant under both transformations. We shall do this completely for the cases $p = 2$ and $p = 3$. The case when $p = 5$ is computationally much larger and we have only partially completed it. Finally, as a contextual issue, we should note that the problem that is presented in this thesis is a reformulation of a group-theoretic problem involving iterated wreath products.

Fix a prime $p$. Define $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{0, \ldots, p-1\}$ to be the ring of integers modulo $p$. Note that $\mathbb{Z}_p$ is a field. Define the set $\mathcal{U} = \{\ (u_1, u_2) \mid u_1, u_2 \in \mathbb{Z}_p\}$, which we call the **point space**. Define the **function space** $\mathcal{F} = \{\ f \mid f : \mathcal{U} \to \mathbb{Z}_p\}$ to be the set of all functions mapping $\mathcal{U}$ into $\mathbb{Z}_p$. On $\mathcal{F}$ one has the operations of pointwise addition of functions and pointwise scalar multiplication of a function by an element of $\mathbb{Z}_p$. For each point $u = (u_1, u_2) \in \mathcal{U}$, we define the characteristic function

$f_u = f_{u_1,u_2} \in \mathcal{F}$ by the rule

$$f_{u_1,u_2}(v_1, v_2) = \begin{cases} 1 & \text{if } (v_1, v_2) = (u_1, u_2) \\ 0 & \text{if } (v_1, v_2) \neq (u_1, u_2). \end{cases}$$

It is clear that for each function $f \in \mathcal{F}$ we have

$$f(v_1, v_2) = \sum_{(u_1,u_2)\in\mathcal{U}} f(u_1, u_2) f_{u_1,u_2}(v_1, v_2).$$

Thus, our function space $\mathcal{F}$ is a vector space of dimension $p^2$ over the field $\mathbb{Z}_p$. The following definition will give us the linear transformations which will be the focus of the rest of the thesis.

DEFINITION 1: Let $\mathcal{B} = \{\, f_{u_1,u_2} \mid (u_1, u_2) \in \mathcal{U}\}$ be the basis of characteristic functions for the function space $\mathcal{F}$. Fix a function $f \in \mathcal{F}$ and for each $(u_1, u_2) \in \mathcal{U}$ let $\alpha_{u_1,u_2} = f(u_1, u_2)$. Then we define the **partial derivative of $f$ with respect to the first component**, denoted $\partial_1(f)$, to be the function $\sum_{u_1,u_2} \beta_{u_1,u_2} f_{u_1,u_2}$ where

$$\beta_{u_1,u_2} = \begin{cases} \alpha_{u_1+1,u_2} & \text{if } u_1 \neq p-1 \\ 0 & \text{if } u_1 = p-1. \end{cases}$$

Similarly, we define the **partial derivative of $f$ with respect to the second component**, denoted $\partial_2(f)$, to be the function $\sum_{u_1,u_2} \beta_{u_1,u_2} f_{u_1,u_2}$ where

$$\beta_{u_1,u_2} = \begin{cases} \alpha_{u_1,u_2+1} & \text{if } u_2 \neq p-1 \\ 0 & \text{if } u_2 = p-1. \end{cases}$$

We now illustrate the above definition as follows. As noted earlier, each function $f \in \mathcal{F}$ has the form

$$\begin{array}{ccccccccc} f & = & \alpha_{0,0}f_{0,0} & + & \alpha_{0,1}f_{0,1} & + & \dots & + & \alpha_{0,p-1}f_{0,p-1} \\ & + & \alpha_{1,0}f_{1,0} & + & \alpha_{1,1}f_{1,1} & + & \dots & + & \alpha_{1,p-1}f_{1,p-1} \\ & + & \alpha_{2,0}f_{2,0} & + & \alpha_{2,1}f_{2,1} & + & \dots & + & \alpha_{2,p-1}f_{2,p-1} \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & + & \alpha_{p-1,0}f_{p-1,0} & + & \alpha_{p-1,1}f_{p-1,1} & + & \dots & + & \alpha_{p-1,p-1}f_{p-1,p-1} \end{array}$$

for unique scalars $\alpha_{0,0}, \alpha_{0,1}, \dots, \alpha_{p-1,p-2}, \alpha_{p-1,p-1} \in \mathbb{Z}_p$. Then we obtain

$$\begin{array}{ccccccccc} \partial_1(f) & = & \alpha_{1,0}f_{0,0} & + & \alpha_{1,1}f_{0,1} & + & \dots & + & \alpha_{1,p-1}f_{0,p-1} \\ & + & \alpha_{2,0}f_{1,0} & + & \alpha_{2,1}f_{1,1} & + & \dots & + & \alpha_{2,p-1}f_{1,p-1} \\ & + & \alpha_{3,0}f_{2,0} & + & \alpha_{3,1}f_{2,1} & + & \dots & + & \alpha_{3,p-1}f_{2,p-1} \\ & \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ & + & \alpha_{p-1,0}f_{p-2,0} & + & \alpha_{p-1,1}f_{p-2,1} & + & \dots & + & \alpha_{p-1,p-1}f_{p-2,p-1} \\ & + & 0f_{p-1,0} & + & 0f_{p-1,1} & + & \dots & + & 0f_{p-1,p-1} \end{array}.$$

The reader should keep in mind that due to the above illustration, we can loosely think of the first derivative operator as shifting the coefficients of a function's representation up. Also, we obtain

$$\begin{array}{ccccccccc} \partial_2(f) & = & \alpha_{0,1}f_{0,0} & + & \dots & + & \alpha_{0,p-1}f_{0,p-2} & + & 0f_{0,p-1} \\ & + & \alpha_{1,1}f_{1,0} & + & \dots & + & \alpha_{1,p-1}f_{1,p-2} & + & 0f_{1,p-1} \\ & + & \alpha_{2,1}f_{2,0} & + & \dots & + & \alpha_{2,p-1}f_{2,p-2} & + & 0f_{2,p-1} \\ & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \vdots \\ & + & \alpha_{p-1,1}f_{p-1,0} & + & \dots & + & \alpha_{p-1,p-1}f_{p-1,p-2} & + & 0f_{p-1,p-1} \end{array}.$$

In a similar way to the first derivative, one can think of the second derivative operator shifting the coefficients of a function's representation to the left. The reader should note that each of the linear transformations $\partial_1, \partial_2$ is nilpotent since the composition $\partial_i^p = \partial_i \circ \partial_i \circ \ldots \circ \partial_i$ is the zero transformation for each $i \in \{1, 2\}$. We shall use the notions of partial derivatives and their respective invariant subspaces for the next important definition.

DEFINITION 2: Let $V$ be a subspace of the function space $\mathcal{F}$. We define $V$ to be **doubly-invariant** if for each function $f \in V$, we have $\partial_1(f) \in V$ and $\partial_2(f) \in V$.

We remind the reader that the ultimate goal is to enumerate and identify all of the doubly-invariant subspaces of $\mathcal{F}$. In Chapters II and III we will describe a systematic method to make this task manageable as well as ensure that we count each subspace exactly once. In Chapters IV, V, and VI, we will apply this method to the cases $p = 2$, $p = 3$, and $p = 5$ respectively.

# CHAPTER II

## HOW TO COMPUTE DOUBLY-INVARIANT SPACES

The first observation we have while working towards our goal is that the function space $\mathcal{F}$ is isomorphic to the vector space consisting of all $p \times p$ matrices with entries from the field $\mathbb{Z}_p$, which we denote as $M_{p\times p}(\mathbb{Z}_p)$. For brevity, we shall denote $M_{p\times p}(\mathbb{Z}_p)$ by $\mathcal{M}$. In particular, we shall use the natural isomorphism

$$\begin{array}{cccccccc} f & = & \alpha_{0,0}f_{0,0} & + & \dots & + & \alpha_{0,p-1}f_{0,p-1} \\ & + & \alpha_{1,0}f_{1,0} & + & \dots & + & \alpha_{1,p-1}f_{1,p-1} \\ & + & \alpha_{2,0}f_{2,0} & + & \dots & + & \alpha_{2,p-1}f_{2,p-1} \\ & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ & + & \alpha_{p-1,0}f_{p-1,0} & + & \dots & + & \alpha_{p-1,p-1}f_{p-1,p-1} \end{array} \mapsto \begin{pmatrix} \alpha_{0,0} & \dots & \alpha_{0,p-1} \\ \alpha_{1,0} & \dots & \alpha_{1,p-1} \\ \alpha_{2,0} & \dots & \alpha_{2,p-1} \\ \vdots & \ddots & \vdots \\ \alpha_{p-1,0} & \dots & \alpha_{p-1,p-1} \end{pmatrix}.$$

We shall index the entries in our matrices using points $u = (u_1, u_2)$ from our point space $\mathcal{U}$. We mention this so that the reader may note that the indices of our entries start from 0. Also, instead of labelling our points using letters like $u_1$ and $u_2$, we shall use the more familiar notation of indices which use $i$ and $j$. From the natural isomorphism it seems natural to make the next definition concerning matrices.

DEFINITION 3: Let $m \in \mathcal{M}$ and write

$$m = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \dots & \alpha_{0,p-1} \\ \alpha_{1,0} & \alpha_{1,1} & \dots & \alpha_{1,p-1} \\ \alpha_{2,0} & \alpha_{2,1} & \dots & \alpha_{2,p-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{p-1,0} & \alpha_{p-1,1} & \dots & \alpha_{p-1,p-1} \end{pmatrix}$$

Then we define **the partial of m with respect to the first axis**, denoted as $\partial_1(m)$, and **the partial of m with respect to the second axis**, denoted as $\partial_2(m)$, by the following:

$$\partial_1(m) = \begin{pmatrix} \alpha_{1,0} & \alpha_{1,1} & \alpha_{1,2} & \dots & \alpha_{1,p-2} & \alpha_{1,p-1} \\ \alpha_{2,0} & \alpha_{2,1} & \alpha_{2,2} & \dots & \alpha_{2,p-2} & \alpha_{2,p-1} \\ \alpha_{3,0} & \alpha_{3,1} & \alpha_{3,2} & \dots & \alpha_{3,p-2} & \alpha_{3,p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{p-1,0} & \alpha_{p-1,1} & \alpha_{p-1,2} & \dots & \alpha_{p-1,p-2} & \alpha_{p-1,p-1} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

$$\partial_2(m) = \begin{pmatrix} \alpha_{0,1} & \alpha_{0,2} & \alpha_{0,3} & \dots & \alpha_{0,p-1} & 0 \\ \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \dots & \alpha_{1,p-1} & 0 \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \dots & \alpha_{2,p-1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{p-2,1} & \alpha_{p-2,2} & \alpha_{p-2,3} & \dots & \alpha_{p-2,p-1} & 0 \\ \alpha_{p-1,1} & \alpha_{p-1,2} & \alpha_{p-1,3} & \dots & \alpha_{p-1,p-1} & 0 \end{pmatrix}.$$

It is clear that if $f \mapsto m$, then $\partial_1(f) \mapsto \partial_1(m)$ and $\partial_2(f) \mapsto \partial_2(m)$. Also, like its function space counterpart, $\partial_i$ is nilpotent since $\partial_i^p$ is the zero transformation for each $i \in \{1, 2\}$. We use the above definition to make another natural definition.

DEFINITION 4: A subspace $V$ of $\mathcal{M}$ is said to be **doubly-invariant** if for each $m \in V$, we have $\partial_1(m) \in V$ and $\partial_2(m) \in V$. Further we define the set

$$\mathcal{V} = \{V \mid \text{V is a doubly-invariant subspace of } \mathcal{M}\}.$$

Thus, we have cast the problem of finding the doubly-invariant subspaces of $\mathcal{F}$ to the problem of finding the doubly-invariant subspaces of $\mathcal{M}$. In order to accomplish such a task, we will need a reasonable way to divide the different doubly-invariant subspaces into disjoint subsets. We will need to develop some more concepts in order to do this.

DEFINITION 5: A mapping $\alpha : \mathcal{U} \to \{0, 1\}$ is a **pattern** if it satisfies the following conditions:

1. For each fixed $i_0 \in \{0, \ldots, p-1\}$, the mapping $\alpha(i_0, j)$ is nonincreasing in $j$. In other words $j_1 < j_2$ implies $\alpha(i_0, j_1) \geq \alpha(i_0, j_2)$.
2. For each fixed $j_0 \in \{0, \ldots, p-1\}$, the mapping $\alpha(i, j_0)$ is nonincreasing in $i$. In other words $i_1 < i_2$ implies $\alpha(i_1, j_0) \geq \alpha(i_2, j_0)$.

We shall denote the set of all patterns by $\mathcal{P}$.

Note that we may use matrices in a very natural way to represent patterns. For instance, take $p = 3$ and define the pattern $\alpha$ by $\alpha(0, 0) = \alpha(1, 0) = \alpha(0, 1) = 1$

and $\alpha(i,j) = 0$ for all other points $(i,j) \in \mathcal{U}$. We can now represent $\alpha$ in the following way:

$$\alpha = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

However, we will be using matrices in this thesis for another purpose. As such, if we represent patterns using matrices, confusion might arise. Thus, we adopt the following notational convention. If $\alpha \in \mathcal{P}$ and $\alpha(i,j) = 1$ we place a dot $\bullet$ in the $(i,j)$-entry of the matrix used to represent $\alpha$. However, if $\alpha(i,j) = 0$, we will still place a zero in the $(i,j)$-entry of the matrix. Using the above example, we represent $\alpha$ as follows:

$$\alpha = \begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

With this convention in hand, we are now able to distinguish between matrices in $\mathcal{M}$ and patterns in $\mathcal{P}$. The patterns will be used to decompose the set $\mathcal{V}$ into a union of disjoint subsets which can then be treated individually. First, we shall define a natural way to associate each pattern to a doubly-invariant subspace of $\mathcal{M}$.

DEFINITION 6: For each pattern $\alpha \in \mathcal{P}$, we define the set $E(\alpha) = \{\ e_{i,j} \mid \alpha(i,j) = 1\}$, where $e_{i,j}$ is the matrix whose $(i,j)$-entry is 1 and all of whose other entries are zeros. Using $E(\alpha)$ we define the subspace $V(\alpha) = \langle E(\alpha) \rangle$ of $\mathcal{M}$ to be the **coefficient pattern subspace for the pattern** $\alpha$.

In order to partition the doubly-invariant subspaces, we will need the notion of one pattern "containing" another. To this end, we define the following partial ordering relation.

DEFINITION 7: Given any patterns $\alpha, \beta \in \mathcal{P}$, we write $\alpha \preceq \beta$ if $\alpha(i,j) \leq \beta(i,j)$ for each point $(i,j) \in \mathcal{U}$. If $\alpha \preceq \beta$ and $\alpha \neq \beta$, then we write $\alpha \prec \beta$. If $\alpha \not\preceq \beta$ and $\beta \not\preceq \alpha$, then we say that $\alpha$ and $\beta$ are not comparable.

In terms of coefficient pattern subgroups, it is clear that $\alpha \preceq \beta$ if and only if $V(\alpha) \subseteq V(\beta)$. This leads us to make a very important definition which will allow us to express the set of doubly-invariant subspaces of matrices as a disjoint union.

DEFINITION 8: Let $\alpha \in \mathcal{P}$ be a pattern and $V$ a doubly-invariant subspace of $\mathcal{M}$. We define $\alpha$ to be **maximal in** $V$ if

1. The coefficient subspace for $\alpha$, namely $V(\alpha)$, is contained in $V$. In symbols, $V(\alpha) \subseteq V$.

2. For each $\beta \in \mathcal{P}$ such that $\alpha \prec \beta$ we have $V(\beta) \not\subseteq V$.

Further, for each $\alpha \in \mathcal{P}$ we define the set

$$\mathcal{V}(\alpha) = \{V \mid V \in \mathcal{V} \text{ and } \alpha \text{ is maximal in } V\}.$$

Before we proceed, let us demonstrate that maximality is unique. This will be of use in the theorem that follows.

LEMMA 9: Suppose that for a doubly-invariant subspace $V$ there exist patterns $\alpha, \beta \in \mathcal{P}$ such that $\alpha$ and $\beta$ are both maximal in $V$. Then $\alpha = \beta$.

Proof. There are two main subcases to prove the result. Namely, either $\alpha$ and $\beta$ are comparable or they are not. Suppose that $\alpha$ and $\beta$ are comparable. Then, by definition 7, if $\alpha \prec \beta$, then $\beta$ would be maximal in $V$ and $\alpha$ would not. Similarly, if $\beta \prec \alpha$, then $\alpha$ would be maximal in $V$ and $\beta$ would not. Therefore, $\alpha = \beta$. Now suppose that $\alpha$ and $\beta$ are not comparable. We shall derive a contradiction. Define the function $\gamma$ by the rule $\gamma(i,j) = \max\{\alpha(i,j), \beta(i,j)\}$ for each $(i,j) \in \mathcal{U}$. We shall demonstrate that $\gamma$ is a pattern. Fix an $i_0 \in \{0, \ldots, p-1\}$ and suppose that $\gamma(i_0, j) = 1$. Then either $\alpha(i_0, j) = 1$ or $\beta(i_0, j) = 1$. Without loss of generality, suppose that $\alpha(i_0, j) = 1$. By the definition of pattern, this would mean that $\alpha(i_0, l) = 1 = \gamma(i_0, l)$ for each $l \in \{0, \ldots, j\}$. Hence, $\gamma$ is nonincreasing in $j$. Similarly, we can show that for a fixed $j_0 \in \{0, \ldots, p-1\}$, $\gamma$ is nonincreasing in $i$. Therefore, $\gamma$ is a pattern. Also, by construction $\alpha(i,j), \beta(i,j) \le \gamma(i,j)$ for each $(i,j) \in \mathcal{U}$. Hence, $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. Since $V(\alpha), V(\beta) \subseteq V$, it is clear that $V(\gamma) \subseteq V$. Since $\alpha$ is maximal in $V$, it must be the case that $\alpha = \gamma$. Similarly, $\beta = \gamma$. Hence, $\alpha = \beta$. This is a contradiction since we assumed that $\alpha$ and $\beta$ were not comparable. ∎

Now we have all the necessary machinery in place to prove that the set of doubly-invariant subspaces of matrices can be written as a disjoint union.

THEOREM 10: $\mathcal{V} = \bigcup_{\alpha \in \mathcal{P}} \mathcal{V}(\alpha)$ and this is a disjoint union.

Proof. Let $V \in \mathcal{V}$. We must demonstrate that there exists a pattern $\alpha \in \mathcal{P}$ such that $V \in \mathcal{V}(\alpha)$. To this end, we define the set $\mathcal{P}_V = \{\alpha \in \mathcal{P} \mid V(\alpha) \subseteq V\}$. Consider the pattern $\alpha_0$ which is defined by $\alpha_0(i,j) = 0$ for all $(i,j) \in \mathcal{U}$. Clearly, $\alpha_0 \in \mathcal{P}_V$ which implies that the set $\mathcal{P}_V$ is non-empty. Since there are only a finite number of patterns in $\mathcal{P}_V$ (indeed the cardinality of $\mathcal{P}$ is $2^{p^2}$ by a simple counting argument), one can pick a maximal pattern under the partial ordering. By Lemma 9, we know this pattern is unique.

Now we must demonstrate that the union is disjoint. Suppose that there exists $V \in \mathcal{V}$ and patterns $\alpha, \beta \in \mathcal{P}$ such that $V \in \mathcal{V}(\alpha) \cap \mathcal{V}(\beta)$. Then both $\alpha$ and $\beta$ are maximal in $V$ which implies that $\alpha = \beta$ by Lemma 9. ∎

Before we proceed to another intuitive result pertaining to $\mathcal{V}(\alpha)$, we need an easy result which will prove useful in some of the results to follow.

LEMMA 11: Fix a point $(i,j) \in \mathcal{U}$ and let $B = \{m^{(1)}, \ldots, m^{(n)}\}$ be a set of matrices such that $m^{(k)}_{i,j} = 0$ for each $k \in \{1, \ldots, n\}$. Then if $m \in \text{span}(B)$ we have $m_{i,j} = 0$.

Proof. Since $m \in \text{span}(B)$, we have that $m = \sum_{k=1}^{n} \alpha_k m^{(k)}$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}_p$. This gives us that $m_{i,j} = \sum_{k=1}^{n} \alpha_k \cdot m^{(k)}_{i,j} = 0$. ∎

We may now prove an intuitive and useful result which will give us our starting point for determining each of the sets $\mathcal{V}(\alpha)$.

LEMMA 12: Let $\alpha \in \mathcal{P}$ be a pattern. Then $V(\alpha) \in \mathcal{V}(\alpha)$.

Proof. We must demonstrate that $V(\alpha)$ is doubly-invariant and that $\alpha$ is maximal in $V(\alpha)$. We shall first prove that $V(\alpha)$ is doubly-invariant. In order to accomplish this we must show that $V(\alpha)$ is invariant under both linear transformations $\partial_1, \partial_2$.

We shall show that $V(\alpha)$ is invariant under $\partial_1$. The argument for $\partial_2$ is similar. Keep in mind that showing that a subspace of a vector space is invariant under some transformation is equivalent to showing that the basis vectors are mapped back into the vector space. In particular, our set of basis vectors is $E(\alpha) = \{ e_{i,j} \mid \alpha(i,j) = 1\}$. Let $e_{i,j} \in E(\alpha)$. If $i = 0$, then $\partial_1(e_{i,j}) = \partial_1(e_{0,j}) = 0_m$ where $0_m$ is the zero matrix. Clearly, $0_m \in V(\alpha)$. If $i \neq 0$, then $\partial_1(e_{i,j}) = e_{i-1,j}$. Thus, in order to complete this part of the proof, we must show that $e_{i-1,j} \in V(\alpha)$. Recall that if $e_{i,j} \in E(\alpha)$, then $\alpha(i,j) = 1$. However, by the definition of pattern, we then have that $\alpha(k,j) = 1$ for each $k \in \{0, \ldots, i\}$. In particular $\alpha(i-1,j) = 1$. This implies that $e_{i-1,j} \in E(\alpha) \subseteq V(\alpha)$. Hence, $V(\alpha)$ is invariant under the linear transformation $\partial_1$.

Now we shall demonstrate that $\alpha$ is maximal in $V(\alpha)$. Clearly, part 1 of Definition 8 is satisfied. To prove that part 2 of the definition is satisfied, we shall argue by contradiction. Suppose to the contrary that there exists a pattern $\beta \in \mathcal{P}$ such that $\alpha \prec \beta$ and that $V(\beta) \subseteq V(\alpha)$. However, this means that $\beta \preceq \alpha$. This is a contradiction. Therefore, $\alpha$ is maximal in $V(\alpha)$. ∎

Now that we have our foundation point for investigating the subspaces in $\mathcal{V}(\alpha)$, we can proceed to our method for counting and computing the subspaces of $\mathcal{V}(\alpha)$. Before proceeding, we shall need a definition. The terminology is inspired from the group theory context from which this problem arises.

DEFINITION 13: Fix a subspace $V \in \mathcal{V}$. For each $i \in \{1, 2\}$, we define **the ith centralizer of V** to be the set $C_i(V)$ consisting of all matrices whose derivatives with respect to the $i$th axis are in $V$. That is, $C_i(V) = \partial_i^{-1}(V)$. Further, we define **the centralizer of V** to be the set

$$C(V) = \partial_1^{-1}(V) \cap \partial_2^{-1}(V) = \{m \in \mathcal{M} \mid \partial_1(m) \in V \text{ and } \partial_2(m) \in V\}.$$

We now justify the terminology of the preceding definition. Fix $i \in \{1, 2\}$. In the original group-theoretic context in which the problem of this thesis arose, the operator $\partial_i$ corresponded to commutation with a certain group element in a group containing $\mathcal{M}$ as a normal subgroup. For each $V \in \mathcal{V}$, the centralizer of that element in the quotient group $\mathcal{M}/V$ corresponds to what we call $C_i(V)$.

The reader should realize that both $C_1(V)$ and $C_2(V)$ are subspaces of $\mathcal{M}$ since $\partial_1$ and $\partial_2$ are linear transformations. Further, from the definition it is clear that if $V_1, V_2 \in \mathcal{V}$, and $V_1 \subseteq V_2$, then $C(V_1) \subseteq C(V_2)$. We also note that if $V \in \mathcal{V}$ then $V \subseteq C(V)$ since $V$ is doubly-invariant by definition. Further we have two important results for determining the centralizer of a doubly-invariant subspace.

LEMMA 14: Fix a subspace $V \in \mathcal{V}$ and fix a point $(i_0, j_0) \in \mathcal{U}$. Suppose that $B = \{m^{(1)}, \ldots, m^{(n)}\}$ is a basis for $V$ such that $m^{(k)}_{i_0,j_0} = 0$. Then if $m \in C(V)$, we have $m_{i_0+1,j_0} = m_{i_0,j_0+1} = 0$.

Proof. We shall prove that $m_{i_0+1,j_0} = 0$. The proof that $m_{i_0,j_0+1} = 0$ follows similarly. Let $a = \partial_1(m) \in V$. Then by definition, $a_{i_0,j_0} = m_{i_0+1,j_0}$. However, note that our basis $B$ satisfies the hypotheses of Lemma 11. Thus, $m_{i_0+1,j_0} = a_{i_0,j_0} = 0$. ∎

COROLLARY 15: Let $V \in \mathcal{V}$ and fix a point $(i_0, j_0) \in \mathcal{U}$. Assume $m \in C(V)$ and that $B = \{m^{(1)}, \ldots, m^{(n)}\}$ is a basis for $V$. Finally, let $m \in C(V)$ such that $m_{i_0,j_0} \neq 0$. Then we have the following:

1. If $i_0 = 0$ and $j_0 \neq 0$, then there exists a matrix $m^{(k)} \in B$ such that $m^{(k)}_{i_0,j_0-1} \neq 0$.
2. If $i_0 \neq 0$ and $j_0 = 0$, then there exists a matrix $m^{(k)} \in B$ such that $m^{(k)}_{i_0-1,j_0} \neq 0$.
3. If $i_0 \neq 0$ and $j_0 \neq 0$ then there exist matrices $m^{(k)}, m^{(l)} \in B$ such that $m^{(k)}_{i_0-1,j_0} \neq 0$ and $m^{(l)}_{i_0,j_0-1} \neq 0$.

The above lemma will be very useful later on when we actually start to compute what the centralizers of the different subspaces in $\mathcal{V}$ look like. This will be the case since it will allow us to know a priori that certain entries in the various matrices in our centralizer are zero.

Next we shall give a theorem which will allow one to directly compute the centralizers of coefficient pattern subspaces. Since the precise statement of this theorem will look obtuse at first glance, we offer the reader some examples. Hopefully this will allow the reader to have a picture in mind when reading the statement of the theorem.

For the sake of illustration, let $p = 5$. Consider the pattern

$$\alpha = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $C_1(V(\alpha)) = V(\alpha_1)$ and $C_2(V(\alpha)) = V(\alpha_2)$ where

$$\alpha_1 = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \alpha_2 = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, $C(V(\alpha)) = V(\alpha_1) \cap V(\alpha_2) = V(\beta)$ where

$$\beta = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Notice where the new dots occur in our centralizer with respect to the original pattern.

They are highlighted using squares:

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \square \\ \bullet & \bullet & \square & 0 & 0 \\ \square & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now consider another pattern $\alpha$:

$$\alpha = \begin{pmatrix} \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $C_1(V(\alpha)) = V(\alpha_1)$ and $C_2(V(\alpha)) = V(\alpha_2)$ where

$$\alpha_1 = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \alpha_2 = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, the $C(V(\alpha)) = V(\alpha_1) \cap V(\alpha_2) = V(\beta)$ where

$$\beta = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Notice where the new dots occur in our centralizer with respect to the original pattern. They are highlighted using squares:

$$\begin{pmatrix} \bullet & \bullet & \bullet & \square & 0 \\ \bullet & \bullet & \square & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ \square & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In general suppose that we are given a pattern $\alpha \in \mathcal{P}$ and wish to determine the centralizer $C(V(\alpha))$ of its coefficient pattern subspace. Suppose further that $C(V(\alpha)) = V(\beta)$ for the appropriate pattern $\beta \in \mathcal{P}$. Notice that if we loosely regard the sides of the matrix and the rows and columns of dots as "walls", then we can compute $\beta$ by simply "filling in the corners" of $\alpha$. We shall make this precise in the next theorem. Through the statement and proof, the reader should keep the previous two examples in mind as a picture of what is happening.

THEOREM 16: For a pattern $\alpha \in \mathcal{P}$, define a function $\beta : \mathcal{U} \to \{0,1\}$ by

$$\beta(i,j) = \begin{cases} 1 & \begin{cases} \text{if } i = 0 \text{ or } \alpha(i-1,j) = 1 \\ \text{and} \\ j = 0 \text{ or } \alpha(i,j-1) = 1 \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\beta$ is a pattern and $C(V(\alpha)) = V(\beta)$.

Proof. We shall first prove that the function $\beta$ is a pattern. To show that $\beta$ is a pattern, we must show that for a fixed $i$, the function $\beta$ is nonincreasing in $j$ as well as that for a fixed $j$, the function $\beta$ is nonincreasing in $i$. We shall only prove the first of these assertions as the second is proven in a similar manner. In order to do this, we must demonstrate that for a fixed $i$, if $\beta(i,j) = 1$ then $\beta(i,l) = 1$ for each $l \in \{0,\ldots,j\}$. To this end, fix a point $(i_0,j_0) \in \mathcal{U}$ and suppose that $\beta(i_0,j_0) = 1$. Due to the definition of $\beta$ we must break up the proof into two cases.

Case 1: Suppose that $j_0 = 0$. Then it is trivially true that $\beta(i_0,l) = 1$ for each $l \in \{0,\ldots,j_0\}$.

Case 2: Suppose that $j_0 \neq 0$. Then $\alpha(i_0,j_0-1) = 1$. By the definition of pattern, this implies that $\alpha(i_0,l) = 1$ for each $l \in \{0,\ldots,j_0-1\}$. Then by definition of $\beta$, we have $\beta(i_0,l) = 1$ for each $l \in \{1,\ldots,j_0\}$. The only point left to consider is $\beta$'s value at the point $(i_0,0)$. There are two subcases. First, if $i_0 = 0$, then, by definition, $\beta(i_0,0) = \beta(0,0) = 1$. Second, if $i_0 \neq 0$, then by definition, $\alpha(i_0-1,0) = 1$. This implies, by the definition of $\beta$, that $\beta(i_0,0) = 1$. In either case, $\beta(i_0,0) = 1$. Thus, it is the case that for each $l \in \{0, \ldots, j_0\}$ we have $\beta(i_0, l) = 1$. This implies that $\beta$ is nonincreasing in $j$.

We have proven that $\beta$ is a pattern. We must now demonstrate that $C(V(\alpha)) = V(\beta)$. First let $m \in C(V(\alpha))$. Fix a point $(i_0, j_0) \in \mathcal{U}$ and suppose that $m_{i_0,j_0} \neq 0$. We shall consider four cases.

Case 1: Suppose that $(i_0, j_0) = (0, 0)$. By definition, $\beta(i_0, j_0) = \beta(0, 0) = 1$. Hence, $e_{i_0,j_0} \in E(\beta)$.

Case 2: Suppose that $i_0 = 0$ and $j_0 \neq 0$. Then by part 1 of Corollary 15, we have $e_{i_0,j_0-1} \in E(\alpha)$. This implies that $\alpha(i_0, j_0 - 1) = 1$. Thus, by definition, $\beta(i_0, j_0) = 1$ which implies that $e_{i_0,j_0} \in E(\beta)$.

Case 3: Suppose that $i_0 \neq 0$ and $j_0 = 0$. Then by part 2 of Corollary 15, we have $e_{i_0-1,j_0} \in E(\alpha)$. This implies that $\alpha(i_0 - 1, j_0) = 1$. Thus, by definition, $\beta(i_0, j_0) = 1$ which implies that $e_{i_0,j_0} \in E(\beta)$.

Case 4: Suppose that $i_0 \neq 0$ and $j_0 \neq 0$. Then by part 3 of Corollary 15, we have $e_{i_0-1,j_0}, e_{i_0,j_0-1} \in E(\alpha)$. This implies that $\alpha(i_0 - 1, j_0) = 1$ and $\alpha(i_0, j_0 - 1) = 1$. Thus, by definition, $\beta(i_0, j_0) = 1$ which implies that $e_{i_0,j_0} \in E(\beta)$.

In all cases, $e_{i_0,j_0} \in E(\beta)$. Write $E = \{\ (i, j) \in \mathcal{U} \mid \beta(i, j) = 1\ \}$. Then we can write $m = \sum_{(k,l)\in E} m_{i,j} \cdot e_{i,j} \in V(\beta)$. Hence, $C(V(\alpha)) \subseteq V(\beta)$.

We must now prove containment in the other direction. To this end, suppose that for some point $(i_0, j_0) \in \mathcal{U}$ the matrix $e_{i_0,j_0} \in V(\beta)$. We must show that $\partial_1(e_{i_0,j_0}) \in V(\alpha)$ and $\partial_2(e_{i_0,j_0}) \in V(\alpha)$. We shall show that $\partial_1(e_{i_0,j_0}) \in V(\alpha)$ as the proof that $\partial_2(e_{i_0,j_0}) \in V(\alpha)$ is similar. By definition, $e_{i_0,j_0} \in V(\beta)$ implies that $\beta(i_0, j_0) = 1$. There are two cases to consider. First, if $i_0 = 0$, then $\partial_1(e_{i_0,j_0}) = \partial_1(e_{0,j_0}) = 0_m$ where $0_m$ is the zero matrix. Clearly, $0_m \in V(\alpha)$. Second, if $i_0 \neq 0$, then $\alpha(i_0 - 1, j_0) = 1$. This implies that $\partial_1(e_{i_0,j_0}) = e_{i_0-1,j_0} \in V(\alpha)$. In either case, $\partial_1(e_{i_0,j_0}) \in V(\alpha)$. Thus, $e_{i_0,j_0} \in V(\beta)$ implies that $e_{i_0,j_0} \in C(V(\alpha))$. Hence, $V(\beta)$'s basis elements are in $C(V(\alpha))$. This implies that $V(\beta) \subseteq C(V(\alpha))$. Therefore, $V(\beta) = C(V(\alpha))$ and we are done. ▮

Theorem 16 gives us not only a useful characterization of the centralizers of the coefficient pattern subspaces, but will also play a pivotal role in determining when the process we use to construct our subspaces terminates. We need just one more main result to obtain said algorithm.

THEOREM 17: Let $W \subseteq V$ where $W \in \mathcal{V}$ and $V \in \mathcal{V}$. Suppose further that $\dim(V) = \dim(W) + 1$. Then $V \subseteq C(W)$.

Proof. Fix $v_0 \in V - W$ and note that $V/W = \{\, c \cdot v_0 + W \mid c \in \mathbb{Z}_p\}$. Fix $i \in \{1, 2\}$. Since both $V$ and $W$ are $\partial_i$-invariant, the linear transformation $\partial_i$ induces a linear transformation on the one-dimensional quotient space $V/W$. Write $u_0 = \partial_i(v_0)$. Hence, there exists a constant $c_0 \in \mathbb{Z}_p$ such that $u_0 + W = c_0 v_0 + W$. Thus, for each $v + W = cv_0 \in V/W$, we have $\partial_i(v + W) = \partial_i(c \cdot v_0) + W = cc_0 \cdot v_0 + W = c_0 \cdot v + W$. Hence, $\partial_i$ acts as multiplication by the scalar $c_0$ on $V/W$. This further implies that $\partial_i^p(v + W) = c_0^p \cdot v + W$. Recall that the linear transformation $\partial_i$ is nilpotent. In particular, from the definition of $\partial_i$, we have that $\partial_i^p = 0$. Thus, for each $v + W \in V/W$, we have $W = 0 + W = \partial_i^p(v) + W = \partial_i^p(v + W) = c_0^p \cdot v + W$.

This implies that $c_0^p = 0$. Since $\mathbb{Z}_p$ is a field, this demonstrates that $c_0 = 0$. Thus, the linear transformation induced by $\partial_i$ acts trivially on the quotient space $V/W$. Hence, $\partial_i(V) \subseteq W$ which gives our result that $V \subseteq C(W)$. ▮

Although the last theorem was a bit abstract in its statement, it will allow us to construct the subspaces of $\mathcal{V}$ iteratively. To this end, consider a pattern $\alpha \in \mathcal{P}$. If $V \in \mathcal{V}(\alpha)$, then there exist matrices $m_1, \ldots, m_n$, such that $V = \langle E(\alpha), m_1, \ldots, m_n \rangle$ where $E(\alpha) \cup \{m_1, \ldots, m_n\}$ is a linearly independent set. Write $V = V_n$ and $V(\alpha) = V_0$. The question becomes how to proceed from $V_0$ to $V_n$ in a sensible way. This is answered using the above theorem. Write $V_1 = \langle V_0, m_1 \rangle$. Then $\dim(V_1) = \dim(V_0) + 1$ and $V_0 \subseteq V_1$. Also, both $V_0$ and $V_1$ are doubly-invariant by assumption. Hence, Theorem 17 yields that $V_1 \subseteq C(V_0)$. By assumption $E(\alpha) \cup \{m_1\}$ is a linearly independent set. Thus, $m_1 \in C(V_0)$ but $m_1 \notin V_0$. We may make a similar argument for $m_2$. Write $V_2 = \langle V_0, m_1, m_2 \rangle$. Then $\dim(V_2) = \dim(V_1) + 1$ and $V_1 \subseteq V_2$. Also, both $V_1$ and $V_2$ are doubly-invariant by assumption. Hence, Theorem 17 yields that $V_2 \subseteq C(V_1)$. By assumption $E(\alpha) \cup \{m_1, m_2\}$ is a linearly independent set. Thus, $m_2 \in C(V_1)$ but $m_2 \notin V_1$. We may make a similar argument for the matrices $m_3$ through $m_n$.

The above paragraph shows how we may build the elements of $\mathcal{V}(\alpha)$ inductively. Write $V_0 = V(\alpha)$. Compute $C(V_0)$. Choose a non-zero matrix $m_1 \in C(V_0)$ and $m_1 \notin V_0$. Write $V_1 = \langle V_0, m_1 \rangle$. Compute $C(V_1)$. Choose a non-zero matrix $m_2 \in C(V_1)$ and $m_1 \notin V_1$. Write $V_2 = \langle V_0, m_1, m_2 \rangle = \langle V_1, m_2 \rangle$. In general, we may proceed from the subspace $V_i$ to the subspace $V_{i+1}$ in a similar manner. Compute

$C(V_i)$. Choose a non-zero matrix $m_{i+1} \in C(V_i)$ and $m_i \notin V_i$. Write $V_{i+1} = \langle V_i, m_{i+1} \rangle$.

The only question in this whole process is whether or not when we proceed from $V_i$ to $V_{i+1}$ that our new subspace is in $\mathcal{V}(\alpha)$. That is, if $V_i \in \mathcal{V}(\alpha)$, how do we know that $V_{i+1} \in \mathcal{V}(\alpha)$? How are we assured that there might not exist a pattern $\beta \in \mathcal{P}$ such that $V_{i+1} \in \mathcal{V}(\beta)$ even though $V_i \in \mathcal{V}(\alpha)$. The answer to this lies in avoiding adding standard basis matrices to our new subspaces other than the ones which we have from our original coefficient pattern subspace. Before the statement of the theorem is given, the reader will be given two examples to keep in mind.

First, suppose that $p = 5$ and that $V \in \mathcal{V}(\alpha)$ where

$$\alpha = \begin{pmatrix} \bullet & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Suppose that using the above outlined method, we choose a matrix $m \in C(V)$ such that $m \notin V$ and form the subspace $W = \langle V, m \rangle$. Now suppose further that

$$e_{3,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in W.$$

Since $W$ is doubly-invariant, this means that all derivatives of $e_{3,2}$ are in $W$ as well. In particular, we can "move" $e_{3,2}$ left until it runs into a wall of the matrix or column of dots. More precisely, note that

$$\partial_2^2(e_{3,2}) = \partial_2(e_{3,1}) = e_{3,0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in W.$$

Now we may "move" $e_{3,0}$ up until we run into the wall of the matrix or a row of dots. More precisely, note that

$$\partial_1(e_{3,0}) = e_{2,0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in W.$$

Thus, if $e_{3,2} \in W$, then $e_{2,0} \in W$. However, since $V(\alpha) \subseteq W$ it is also the case that $e_{0,0}, e_{1,0}, e_{0,1} \in W$. Notice though, that this means that $V(\beta) \subseteq W$ where

$$\beta = \begin{pmatrix} \bullet & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $\alpha \prec \beta$. Hence, $W \notin \mathcal{V}(\alpha)$.

Again, suppose that $p = 5$ and that $V \in \mathcal{V}(\alpha)$ where

$$\alpha = \begin{pmatrix} \bullet & \bullet & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Suppose that using the above outlined method, we choose a matrix $m \in C(V)$ such that $m \notin V$ and form the subspace $W = \langle V, m \rangle$. Now suppose further that

$$e_{3,4} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in W.$$

Since $W$ is doubly-invariant, this means that all derivatives of $e_{3,4}$ are in $W$ as well. In particular, we can "move" $e_{3,4}$ left until it runs into a wall of the matrix or column of dots. More precisely, note that

$$\partial_2^3(e_{3,4}) = e_{3,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in W.$$

Now we may "move" $e_{3,1}$ up until we run into the wall of the matrix or a row of dots. More precisely, note that

$$\partial_1(e_{3,1}) = e_{2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in W.$$

Thus, if $e_{3,4} \in W$, then $e_{2,1} \in W$. However, since $V(\alpha) \subseteq W$ it is also the case that $e_{0,0}, e_{1,0}, e_{2,0}, e_{3,0}, e_{0,1}, e_{1,1} \in W$. Notice though, that this means that $V(\beta) \subseteq W$ where

$$\beta = \begin{pmatrix} \bullet & \bullet & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $\alpha \prec \beta$. Hence, $W \notin \mathcal{V}(\alpha)$.

The following theorem shall make precise the above illustrations. We will take some arbitrary standard basis matrix and "move" it around until we have evidence that our new subspace is contained in a larger pattern subspace.

THEOREM 18: Fix a point $(i_0, j_0) \in \mathcal{U}$, a pattern $\alpha \in \mathcal{P}$, and a subspace $V \in \mathcal{V}$. Suppose that $V(\alpha) \subseteq V$ and $e_{i_0,j_0} \notin E(\alpha)$ and $e_{i_0,j_0} \in V$. Then $V \notin \mathcal{V}(\alpha)$.

Proof. In order to prove the theorem we shall construct a new pattern $\beta$ and show that $V(\beta) \subseteq V$ and $\alpha \prec \beta$. This will give us our desired result. Since $V \in \mathcal{V}$, we know that $\{\partial_1^{i_0}(e_{i_0,j_0}), \partial_1^{i_0-1}(e_{i_0,j_0}), \ldots, \partial_1(e_{i_0,j_0}), e_{i_0,j_0}\} = \{e_{0,j_0}, e_{1,j_0} \ldots e_{i_0,j_0}\} \subseteq V$. In other words, for each $k \in \{0, \ldots, i_0\}$ we have $e_{k,j_0} \in V$. Once again, since $V \in \mathcal{V}$, we know that for each $k \in \{0, \ldots, i_0\}$ we have $\{\partial_2^{j_0}(e_{k,j_0}), \partial_2^{j_0-1}(e_{k,j_0}), \ldots, \partial_2(e_{k,j_0}), e_{k,j_0}\} = \{e_{k,0}, e_{k,1} \ldots e_{k,j_0}\} \subseteq V$. Thus, we have $E = \{\ e_{k,l} \mid 0 \leq k \leq i_0 \text{ and } 0 \leq l \leq j_0\ \} \subseteq V$. We shall now define our new pattern $\beta$ in the following manner:

$$\beta(i,j) = \begin{cases} 1 & \begin{cases} \text{if } 0 \leq i \leq i_0 \text{ and } 0 \leq j \leq j_0 \\ \text{or} \\ \alpha(i,j) = 1 \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

We must still demonstrate that $\beta$ is a pattern. Note that if $\beta$ is a pattern, then $\alpha \prec \beta$ and $V(\beta) \subseteq V$ and our result follows. However, it immediately follows that $\beta$ is a pattern due to the construction of set the $E$ and the fact that $\alpha$ is pattern. ∎

From the above theorem, we see that there is a bit of a hitch in our algorithm to construct the subspaces in $\mathcal{V}(\alpha)$ for some pattern $\alpha \in \mathcal{P}$. That is, in order to move from $V_i$ to $V_{i+1}$, we must choose a matrix $m_i \in C(V_i)$ where $m_i \notin V$. However, we must ensure that if $e_{i,j} \in V_{i+1}$, then $e_{i,j} \in V_i$. Otherwise, Theorem 18 tells us that there exists a pattern $\beta \in \mathcal{P}$ such that $\alpha \prec \beta$ and $V_{i+1} \in \mathcal{V}(\beta)$.

Finally, we may also use the above theorem to state when the algorithm terminates. Consider a pattern $\alpha \in \mathcal{P}$ and let $V \in \mathcal{V}(\alpha)$. If $\dim(C(V)) = \dim(V)+1$, then there is no matrix $m$ such that $m \in C(V)$, $m \notin V$, and $\langle V, m \rangle \in \mathcal{V}(\alpha)$. The reason is as follows. Write $W = \langle V, m \rangle$. Then, $\dim(C(V)) = \dim(W)$ and, by construction, $W \subseteq C(V)$. Thus, $W = C(V)$. Now recall that $V(\alpha) \subseteq V$ which implies that $C(V(\alpha)) \subseteq C(V) = W$. Use theorem 16 to define the pattern $\beta \in \mathcal{P}$, by $V(\beta) = C(V(\alpha))$. From theorem 16, we know that $\alpha \prec \beta$. Hence, $V \in \mathcal{V}(\beta)$. Thus, our process terminates when $\dim(C(V)) = \dim(V) + 1$.

We now write down the algorithm for computing all of the elements of $\mathcal{V}$.

1. Choose a pattern $\alpha \in \mathcal{P}$. Let $i = 0$. Let $V_0 = \mathcal{V}(\alpha)$.

2. Compute $C(V_i)$.

3. If $\dim(C(V)) - \dim(V) \leq 1$, then stop.

4. Choose $m_{i+1} \in C(V_i) - V_i$ such that if $e_{i,j} \in \langle V_i, m_{i+1} \rangle$, then $e_{i,j} \in V_i$.

5. Let $V_{i+1} = \langle V_i, m_{i+1} \rangle$.

6. Increment $i$ and go to 2.

In this way, we will be able to compute all of the subspaces of $\mathcal{V}$. However, the difficult part will be avoiding adding new standard basis vectors to our subspaces. This is the topic of the next chapter.

We finish this chapter by giving some additional definitions which will be useful in recording information about the subspaces of $\mathcal{V}$.

DEFINITION (19): For a pattern $\alpha \in \mathcal{P}$ and a nonnegative integer $\ell$, define **the set of subspaces for the pattern $\alpha$ at level $\ell$**, denoted $\mathcal{V}(\alpha)_\ell$, to be the set of subspaces $V \in \mathcal{V}(\alpha)$ such that $\dim(V) = \dim(V(\alpha)) + \ell$. In symbols, $\mathcal{V}(\alpha)_\ell = \{\, V \in \mathcal{V}(\alpha) \mid \dim(V) = \dim(V(\alpha)) + \ell \,\}$.

Clearly, for a pattern $\alpha \in \mathcal{P}$, we have $\mathcal{V}(\alpha)_0 = \{V(\alpha)\}$, and we have the disjoint union $\mathcal{V}(\alpha) = \bigcup_{\ell=0}^{\infty} \mathcal{V}(\alpha)_\ell$. Also, since $\mathcal{M}$ is finite dimensional, there exists a unique nonnegative integer $\mathrm{ht}(\alpha)$ dependent on $\alpha$ such that $\mathcal{V}(\alpha)_\ell$ is non-empty for $\ell \leq \mathrm{ht}(\alpha)$ and $\mathcal{V}(\alpha)_\ell$ is empty for $\ell > \mathrm{ht}(\alpha)$. We define $\mathrm{ht}(\alpha)$ to be the **height** of the pattern $\alpha$. Thus, we can write

$$\mathcal{V}(\alpha) = \bigcup_{\ell=0}^{\mathrm{ht}(\alpha)} \mathcal{V}(\alpha)_\ell,$$

and this last union is disjoint.

# CHAPTER III

# SUBSPACES NOT CONTAINING STANDARD BASIS VECTORS

In Chapter II, we noted that avoiding standard basis matrices is an essential part of our algorithm for enumerating and identifying the doubly-invariant subspaces of $\mathcal{M}$. We need a systematic way to do this. Recall that we can regard the set of $p \times p$ matrices as a vector space of dimension $p^2$ over its associated field. In this section we will simply look at the space of $n$-tuples over the field $\mathbb{Z}_p$. We will develop a convenient way to represent subspaces that do not contain standard basis vectors. This will, in turn, give us a convenient way to represent elements of those subspaces as well as allow us to easily count the number of such subspaces.

Let $p$ be a prime and let $K$ be the finite field of order $p$. Fix an integer $n \geq 2$. Let $V$ be the set of all ordered $n$-tuples with components from $K$. Thus we write $V = \{ \ (v_1, \dots, v_n) \ | v_i \in K \ \}$. For each index $i \in \{1, \dots, n\}$, let $\overline{e}_i = (0 \dots, 0, 1, 0 \dots 0)$ be the vector whose $i$th component is 1 and all of whose other components are 0. Thus, $\{\overline{e}_1, \dots, \overline{e}_n\}$ is an ordered basis for the $n$-dimensional vector space $V$ over $K$. We refer to each $\overline{e}_i$ as a standard basis vector. We wish to investigate the subspaces of $V$ which do not contain any standard basis vectors. In particular, for a given positive integer $m$ such that $m < n$, we want to count the number of such subspaces of dimension $m$ in $V$, as well as obtain a convenient basis for each. We mention that this convenient basis will be obtained using ideas which are very similar to the notion of reduced row echelon form for a matrix . To this end, we will make use of the following useful functions $\alpha$ and $\nu_i$.

DEFINITION 1: Let $\overline{v} = (v_1, \ldots, v_n) \in V$. We define

$$\alpha(\overline{v}) = \begin{cases} \min(\{i \mid v_i \neq 0\}) & \text{if } \overline{v} \neq \overline{0} \\ 0 & \text{if } \overline{v} = \overline{0}. \end{cases}$$

For $1 \leq i \leq n$, we define $\nu_i(\overline{v}) = v_i$.

It should be noted that $\nu_i$ is a linear function. Using $\nu_i$ we note that $\nu_j(\overline{e}_i) = \delta_{i,j}$ where $\delta_{i,j}$ is the familiar Kronecker delta function,

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

What follows are the main definitions of this section and the results we need in order to proceed.

DEFINITION 2: A subspace $W$ of $V$ is a **good subspace** if $\{\overline{e}_1, \ldots, \overline{e}_n\} \cap W = \emptyset$.

DEFINITION 3: A **set of leading positions**, $T = \{t_1, \ldots, t_m\}$, is a strictly increasing sequence of indices from the set $\{1, \ldots, n-1\}$.

DEFINITION 4: A **good vector** $\overline{v}$ is any non-zero vector having these properties:

1. $\overline{v} \notin \{\overline{e}_1 \ldots \overline{e}_n\}$

2. If $\alpha(\overline{v}) = t$, then $\nu_t(\overline{v}) = 1$.

DEFINITION 5: Let $m$ be a positive integer such that $m < n$, and let $T = \{t_1, \ldots, t_m\}$ be a set of leading positions. A **canonical basis corresponding to** $T$ is an ordered set of good vectors $\{\overline{v}_1, \ldots, \overline{v}_m\}$ such that:

1. $\alpha(\overline{v}_i) = t_i$.

2. $\nu_{t_j}(\overline{v}_i) = \delta_{i,j}$.

In a bit we will justify our use of the term canonical as well as list some of the nice properties of such bases. However, the first matter to attend to is to show that every good subspace of $V$ has a canonical basis corresponding to some set of leading positions.

THEOREM 6: Let $m$ be any positive integer such that $m < n$. If $W$ is an $m$-dimensional good subspace of $V$, then there exists a set of leading positions, $T = \{t_1, \ldots, t_m\}$, and a canonical basis $B = \{\overline{v}_1, \ldots, \overline{v}_m\}$ for $W$ corresponding to $T$.

Proof. Since $W$ is an $m$-dimensional subspace, it has some basis $B' = \{\overline{w}_1, \ldots, \overline{w}_m\}$. Let $M' = [a'_{i,j}]$ be the $m \times n$ matrix whose rows are the vectors from $B'$. More precisely $a'_{i,j} = w_{i,j}$ where $w_{i,j}$ is the $j$th entry of the vector $\overline{w}_i$, i.e. $a'_{i,j} = \nu_j(\overline{w}_i)$. By elementary linear algebra, there is another $m \times n$ matrix $M = [a_{i,j}]$ where $M$ is the reduced row echelon form for $M'$. Let $B = \{\overline{v}_1, \ldots, \overline{v}_m\}$ be the set of row vectors from $M$ such that $v_i$ is the $i$th row vector from $M$. By elementary linear algebra, we know that the row space of a matrix is invariant under elementary row operations. Further, the non-zero rows in any matrix in reduced row echelon form are always linearly independent. Hence, $B$ is a basis for $W$. Further, $B \cap \{\overline{e}_1, \ldots, \overline{e}_n\} = \emptyset$ by assumption since $W$ is a good subspace. Define $T = \{t_1, \dots, t_m\}$ by $t_i = \alpha(\overline{v}_i)$. Since $\overline{e}_n \notin B$, we know that $t_m \leq n-1$. Further, from the definition of reduced row echelon form, we have that $\nu_{t_j}(\overline{v}_i) = \delta_{i,j}$ and that if $\alpha(\overline{v}_i) = t_i$, then $t_j > t_i$ when $j > i$. Thus, $T$ is a set of leading positions and $B$ is a canonical basis corresponding to $T$. ▮

We now know that every good subspace has a canonical basis corresponding to some set of leading positions. What we now wish to establish is the converse, namely, that if you have some set of leading positions and a basis which corresponds to it, then the span of that basis is a good subspace. Before we do this, we will need an intermediate result which is useful in its own right for computations we do down the road.

LEMMA 7: Let $W$ be a subspace of $V$, let $T = \{t_1, \dots, t_m\}$ be a set of leading positions, and suppose that that $W$ has a basis $B = \{\overline{w}_1, \dots, \overline{w}_m\}$ such that $\nu_{t_j}(\overline{w}_i) = \delta_{i,j}$, then $\overline{x} = \sum_{i=1}^{m} \nu_{t_i}(\overline{x})\overline{w}_i$.

Proof. The fact that $\overline{x} \in W$ implies that $\overline{x} = \sum_{i=1}^{m} \alpha_i \overline{w}_i$ for some set of coefficients $\alpha_1, \dots, \alpha_m \in K$. By our assumption on $B$, we have that $\nu_{t_j}(\overline{w}_i) = \delta_{i,j}$ for each pair of indices $i, j \in \{1, \dots, m\}$. Hence for each $j \in \{1, \dots, m\}$, using the linearity of $\nu_{t_j}$ we obtain

$$\nu_{t_j}(\overline{x}) = \sum_{i=1}^{m} \alpha_i \nu_{t_j}(\overline{w}_i) = \sum_{i=1}^{m} \alpha_i \delta_{i,j} = \alpha_j.$$

For use in later chapters, the reader should note that Lemma 7 did not assume that $W$ is good subspace. We may now proceed to the desired theorem.

THEOREM 8: Let $T = \{t_1, \ldots, t_m\}$ be a set of leading positions and let $B = \{\overline{w}_1, \ldots, \overline{w}_m\}$ be a canonical basis corresponding to $T$. Write $W = \text{span}(B)$. Then:

1. If $\overline{x} \in W$, then $\overline{x} = \sum_{i=1}^{m} \nu_{t_i}(\overline{x})\overline{w}_i$.

2. $W$ is a good subspace of $V$.

3. If $B' = \{\overline{v}_1, \ldots, \overline{v}_m\}$ is another canonical basis corresponding to $T$ and $B \neq B'$, then $W \neq \text{span}(B')$.

Proof. 1. This follows immediately by Lemma 7 and the definition of a canonical basis.

2. We must show that $W \cap \{\overline{e}_1, \ldots, \overline{e}_n\} = \emptyset$. To this end, we assume instead that $\overline{e}_j \in W$ for some index $j \in \{1, \ldots, n\}$, and work for a contradiction. By Lemma 7 we have $\overline{e}_j = \sum_{i=1}^{m} \nu_{t_i}(\overline{e}_j)\overline{w}_i$. For each $i \in \{1, \ldots, m\}$, recall that $\nu_{t_i}(\overline{e}_j) = \delta_{t_i,j}$. If $j \notin T$, then $\nu_{t_i}(\overline{e}_j) = 0$ for each $i \in \{1, \ldots, m\}$, which implies that $\overline{e}_j = \sum_{i=1}^{m} \nu_{t_i}(\overline{e}_j)\overline{w}_i = \sum_{i=1}^{m} 0 \cdot \overline{w}_i = \overline{0}$. This is a contradiction. Hence, $j \in T$ which implies that $j = t_k$ for some $k \in \{1, \ldots, m\}$. Thus, $\overline{e}_j = \overline{e}_{t_k} = \sum_{i=1}^{m} \nu_{t_i}(\overline{e}_{t_i})\overline{w}_i = \overline{w}_i$. However, this contradicts the assumption that $\overline{w}_i$ is a good vector. Therefore, $\overline{e}_j \notin W$ which implies that $W$ is a good subspace and part 2 is proved.

3. The fact that $B \neq B'$ implies that there is an index $j \in \{1, \ldots, n\}$ such that $\overline{w}_j \neq \overline{v}_j$. Suppose for a contradiction that $W = \text{span}(B')$. Then $\overline{v}_j \in W$ and by part 1 of the theorem and the fact that $B'$ is a canonical basis corresponding to $T$, we have $\overline{v}_j = \sum_{i=1}^{m} \nu_{t_i}(\overline{v}_j)\overline{w}_i = \overline{w}_j$. This is a contradiction. Hence, part 3 is proved. ▌

Thanks to Theorems 6 and 8, we are justified in describing as canonical the bases defined by Definition 5. Thus, we may speak of the canonical basis of any good subspace. Finally, we also have a way to construct all canonical bases for all good subspaces of $V$:

1. Fix a positive integer $m$ such that $m < n$.
2. Choose a set of leading positions, $T = \{t_1, \ldots, t_m\}$.
3. Choose a basis $B = \{\overline{v}_1, \ldots, \overline{v}_m\}$ such that $\overline{v}_i$ is a good vector and such that $\nu_{t_j}(\overline{v}_i) = \delta_{i,j}$.
4. Repeat step 3 for each set of leading positions of size $m$.
5. Repeat steps 2–4 for each $m$ such that $1 \leq m \leq n - 1$.

Notice that step 3 is accomplished by setting the $t_i$th component of the vector $\overline{v}_i$ equal to 1, setting all components before the $t_i$th entry equal to zero, and choosing arbitrary values for all other components which are not the $t_j$th components and making sure that there is at least one non-zero entry past the $t_i$th entry. Notice that constructing bases in this fashion allows us to easily count how many there are for a given set of leading positions $T = \{t_1, \ldots, t_m\}$. Since each canonical basis corresponds to a unique good subspace and vice versa, we are thus able to count how many good subspaces there are associated with $T$.

DEFINITION 9: Let $W$ be a good subspace of $V$ and $B = \{\overline{v}_1, \ldots, \overline{v}_m\}$ its canonical basis corresponding to some set of leading positions $T = \{t_1, \ldots, t_m\}$ which is guaranteed to exist by Theorem 6. Then $W$ is said to be **associated** to $T$.

We note that Definition 9 is unambiguous thanks to Theorems 6 and 7. We now use this definition and the paragraph preceding it to form the following theorem.

THEOREM 10: Fix a positive integer $m$ such that $m < n$, and let $T = \{t_1, \ldots, t_m\}$ be a set of leading positions. Then the number of good subspaces of $V$ that are associated to $T$, denoted $g(T)$, is

$$\prod_{i=1}^{m}(p^{n-m-t_i+i} - 1).$$

Proof. We must count how many ways there are to choose a canonical basis $B = \{\overline{v}_1, \ldots, \overline{v}_m\}$ corresponding to the set of leading positions $T$. This amounts to counting the number of different ways there are to choose the $i$th vector. Since the vector $\overline{v}_i$ corresponds to the number $t_i$, we have that the first $t_i - 1$ entries of the $i$th vector are zero. Also, since $B$ is a canonical basis by assumption, entry $t_i$ is 1, and entries $t_{i+1}, \ldots, t_m$ are all zero. This leaves $n - t_i - (m - i)$ entries left to fill. We may arbitrarily choose their values except for the case in which all are zero. Hence, we have $p^{n-t_i-(m-i)} - 1$ choices for the $i$th vector. Thus, multiplying the choices we have for each vector together, we find that we have

$$\prod_{i=1}^{m}(p^{n-m-t_i+i} - 1).$$

choices total for our basis and the theorem is proved. ∎

Table 3.1: Illustration of Theorem 10

| $n$ | $m$ | $T$ | $g(T)$ |
|---|---|---|---|
| 2 | 1 | {1} | $p-1$ |
| 3 | 1 | {1} | $p^2-1$ |
| 3 | 1 | {2} | $p-1$ |
| 3 | 2 | {1,2} | $(p-1)(p-1)$ |
| 4 | 1 | {1} | $p^3-1$ |
| 4 | 1 | {2} | $p^2-1$ |
| 4 | 1 | {3} | $p-1$ |
| 4 | 2 | {1,2} | $(p^2-1)(p^2-1)$ |
| 4 | 2 | {1,3} | $(p^2-1)(p-1)$ |
| 4 | 2 | {2,3} | $(p-1)(p-1)$ |
| 4 | 3 | {1,2,3} | $(p-1)(p-1)(p-1)$ |

Table 3.1 illustrates Theorem 10 under various inputs for the dimension of the ambient vector space $n$, the dimension of the subspace $m$, and the set of leading positions $T$.

# CHAPTER IV

# THE RESULTS FOR THE PRIME 2

For the prime $p = 2$, the only interesting pattern is

$$\alpha = \begin{pmatrix} \bullet & 0 \\ 0 & 0 \end{pmatrix}.$$

We want to list the subspaces in the set $\mathcal{V}(\alpha)$. To this end, we define the basis $B_0 = E(\alpha)$ and the subspace $V_0 = V(\alpha) = \langle B_0 \rangle$. Theorem 16 from Chapter II demonstrates that $C(V_0) = V(\beta)$ where

$$\beta = \begin{pmatrix} \bullet & \bullet \\ \bullet & 0 \end{pmatrix}.$$

Since $\dim(C(V_0)) - \dim(V_0) = 2$, we must choose a matrix $m_1 \in C(V_0) - V_0$ such that for each point $(i,j) \in \mathcal{U}$, if $e_{i,j} \in \langle V_0, m_1 \rangle$, then $e_{i,j} \in V_0$. Hence, we want the 1-dimensional good subspaces of $\mathbb{Z}_2 \times \mathbb{Z}_2$. From the construction of such spaces in the previous chapter, we know that there is only one. This is generated by the vector $(1,1)$. To this end, we define

$$m_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$B_1 = E(\alpha) \cup \{m_1\}$, and $V_1 = \langle B_1 \rangle$. Note that $B_1$ is a basis of the subspace $V_1$ that satisfies the hypotheses of Lemma 7 from Chapter III.

Table 4.1: Results For The Pattern $\alpha$

| $\mathrm{ht}(\alpha)$ | $\lvert\mathcal{V}(\alpha)_0\rvert$ | $\lvert\mathcal{V}(\alpha)_1\rvert$ | $\lvert\mathcal{V}(\alpha)\rvert$ |
|---|---|---|---|
| 1 | 1 | 1 | 2 |

Let us now compute the centralizer of $V_1$. Each element of $C(V_1)$ has form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} \\ x_{1,0} & x_{1,1} \end{pmatrix}.$$

Note that $x \in C(V_1)$ if and only if $\partial_1(x) \in V_1$ and $\partial_2(x) \in V_1$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & 0 \\ x_{1,1} & 0 \end{pmatrix}.$$

Suppose $\partial_1(x) \in V_1$, and consider the expansion of $\partial_1(x)$ with respect to the basis $B_1$. By Lemma 7 in Chapter III the coefficient on $m_1$ in the expansion of $\partial_1(x)$ must be 0. This would imply that $x_{1,1} = 0$ and $\dim(C(V_1)) - \dim(V_1) = 1$. Thus our algorithm terminates and we are finished with this pattern. A summary is given in the table below.

For any other pattern $\gamma \in \mathcal{P}$ such that $\gamma \neq \alpha$ it is the case that $\dim(C(V(\gamma))) - \dim(V(\gamma)) \leq 1$. The reader can verify this by applying Theorem 16 to any pattern in question. However, there is an easier way to see this using the language we used in Chapter II to motivate Theorem 16. The reader should note that each of these patterns has at most one "corner". These patterns are:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \bullet & \bullet \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \bullet & 0 \\ \bullet & 0 \end{pmatrix}, \begin{pmatrix} \bullet & \bullet \\ \bullet & 0 \end{pmatrix}, \begin{pmatrix} \bullet & \bullet \\ \bullet & \bullet \end{pmatrix}.$$

For each of these five patterns $\gamma$, it is the case that $\mathcal{V}(\gamma) = \{V(\gamma)\}$. Thus, when $p = 2$, there are 7 doubly-invariant subspaces of $\mathcal{M}$. That concludes the results for the case $p = 2$.

# CHAPTER V

# THE RESULTS FOR THE PRIME 3

There are only two interesting patterns in the case $p = 3$. As we shall see later, all of the others are similar to the patterns in the $p = 2$ case. The first interesting pattern is

$$\alpha_1 = \begin{pmatrix} \bullet & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We want to list the subspaces in the set $\mathcal{V}(\alpha_1)$. To this end, we define the basis $B_0 = E(\alpha_1)$ and the subspace $V_0 = V(\alpha_1) = \langle B_0 \rangle$. Theorem 16 from Chapter II demonstrates that $C(V_0) = V(\beta_1)$ where

$$\beta_1 = \begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Define the basis $B_0' = E(\beta_1)$ and note that $V(\beta_1) = \langle B_0' \rangle$. Since $\dim(C(V_0)) - \dim(V_0) = 2$, we must choose a matrix $m_1 \in C(V_0) - V_0$ such that for each point $(i, j) \in \mathcal{U}$, if $e_{i,j} \in \langle V_0, m_1 \rangle$, then $e_{i,j} \in V_0$. Hence, we want the 1-dimensional good subspaces of $\mathbb{Z}_3 \times \mathbb{Z}_3$. From the construction of such spaces in Chapter III, we know that there are two. These are generated by the vectors $(1, 1)$ and $(1, 2)$. Fix any scalar $0 \neq c_1 \in \mathbb{Z}_3$ and define

$$m_1 = \begin{pmatrix} 0 & c_1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$B_1 = E(\alpha_1) \cup \{m_1\}$, and $V_1 = \langle B_1 \rangle$. Note that $B_1$ is a basis of the subspace $V_1$ that satisfies the hypotheses of Lemma 7 from Chapter III.

Let us now compute the centralizer of $V_1$. By Lemma 14 in Chapter II, each element of $C(V_1)$ has form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & 0 \\ x_{2,0} & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_1)$ if and only if $\partial_1(x) \in V_1$ and $\partial_2(x) \in V_1$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & 0 \\ x_{2,0} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & 0 \\ x_{1,1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_1$ if and only if

$$x_{1,1} = c_1 \cdot x_{2,0},$$

and that $\partial_2(x) \in V_1$ if and only if

$$x_{0,2} = c_1 \cdot x_{1,1}.$$

Hence, $x \in C(V_1)$ if and only if

$$x_{1,1} = c_1 \cdot x_{2,0}$$

$$x_{0,2} = c_1^2 \cdot x_{2,0}.$$

Therefore, if we define

$$v_1 = \begin{pmatrix} 0 & 0 & c_1^2 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $B_1' = B_0' \cup \{v_1\}$, then $C(V_1) = \langle B_1' \rangle$. Since $\dim(C(V_1)) - \dim(V_1) = 2$, we must choose a matrix $m_2 \in C(V_1) - V_1$ such that if $e_{i,j} \in \langle V_1, m_2 \rangle$, then $e_{i,j} \in V_1$. If the (1,0)-entry of $m_2$ is non-zero, we can subtract off appropriate multiples of $m_1$ to obtain a new matrix whose (1,0)-entry is 0. We may do this since the (1,0)-entry of $m_1$ is 1. Hence, we may assume without loss of generality that the (1,0)-entry of $m_2$ is 0. Consider the expansion of $m_2$ with respect to the basis $B_1'$. If the coefficient on $v_1$ in the expansion of $m_2$ is 0, then the only non-zero entry of $m_2$ is the (0,1)-entry. However, that would mean $e_{0,1} \in \langle V_1, m_2 \rangle$ even though $e_{0,1} \notin V_1$. Thus, the coefficient on $v_1$ in the expansion of $m_2$ must be non-zero. Since we may scale the coefficient on $v_1$, we shall choose said coefficient to be 1. Hence, we fix $c_2 \in \mathbb{Z}_3$ and define the matrix

$$m_2 = \begin{pmatrix} 0 & c_2 & c_1^2 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

Table 5.1: Results For The Pattern $\alpha_1$

| $\mathrm{ht}(\alpha_1)$ | $\lvert\mathcal{V}(\alpha_1)_0\rvert$ | $\lvert\mathcal{V}(\alpha_1)_1\rvert$ | $\lvert\mathcal{V}(\alpha_1)_2\rvert$ | $\lvert\mathcal{V}(\alpha_1)\rvert$ |
|---|---|---|---|---|
| 2 | 1 | 2 | 6 | 9 |

$B_2 = B_1 \cup \{m_2\}$, and $V_2 = \langle B_2 \rangle$. Note that we may have $c_2 = 0$ without worrying about having a standard basis vector in our span. Note that $B_2$ is a basis for the subspace $V_2$ that satisfies the hypotheses of Lemma 7 in Chapter III.

Let us now compute the centralizer of $V_2$. By Lemma 14 in Chapter II, each element of $C(V_2)$ has form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & 0 \end{pmatrix}.$$

Note that $x \in C(V_2)$ if and only if $\partial_1(x) \in V_2$ and $\partial_2(x) \in V_2$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & 0 \\ x_{1,1} & x_{1,2} & 0 \\ x_{2,1} & 0 & 0 \end{pmatrix}.$$

Suppose $\partial_1(x) \in V_2$, and consider the expansion of $\partial_1(x)$ with respect to the basis $B_2$. By Lemma 7 in Chapter III the coefficient on $m_2$ in the expansion of $\partial_1(x)$ must be 0. This would imply that $x_{2,1} = x_{1,2} = 0$ and $\dim(C(V_2)) - \dim(V_2) = 1$. Thus our algorithm terminates and we are finished with this pattern. A summary is given in table 5.1. Recall that we had 2 choices for $c_1$ and 3 choices for $c_2$.

The second pattern is even more interesting that the first. We define the pattern

$$\alpha_2 = \begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We want to list the subspaces in the set $\mathcal{V}(\alpha_2)$. To this end, we define the basis $B_0 = E(\alpha_2)$ and the subspace $V_0 = V(\alpha_2) = \langle B_0 \rangle$. Theorem 16 from Chapter II demonstrates that $C(V_0) = V(\beta_2)$ where

$$\beta_2 = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & 0 \\ \bullet & 0 & 0 \end{pmatrix}.$$

Define the basis $B_0' = E(\beta_2)$ and note that $V(\beta_2) = \langle B_0' \rangle$. Since $\dim(C(V_0)) - \dim(V_0) = 3$, we must choose a matrix $m_1 \in C(V_0) - V_0$ such that for each point $(i,j) \in \mathcal{U}$, if $e_{i,j} \in \langle V_0, m_1 \rangle$, then $e_{i,j} \in V_0$. Hence, we want the 1-dimensional good subspaces of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. From the construction of such spaces in Chapter III, we know that there are ten such subspaces. These are generated by the vectors $(0,1,1)$, $(0,1,2)$, $(1,0,1)$, $(1,0,2)$, $(1,1,0)$, $(1,1,1)$, $(1,1,2)$, $(1,2,0)$, $(1,2,1)$, and $(1,2,2)$. This count agrees with the table at the end of Chapter III. Note that such subspaces are generated by vectors of two different forms. The first form is $(0,1,c_1)$ where $c_1 \neq 0$. The second form is $(1,c_1,c_2)$ where $(c_1,c_2) \neq (0,0)$. As there are two different forms for the vectors, we shall break up our discussion into two cases.

Case 1: Fix $c_1 \in \mathbb{Z}_3$ and define

$$m_1 = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$B_1 = E(\alpha_2) \cup \{m_1\}$, and $V_1 = \langle B_1 \rangle$. Note that $B_1$ is basis for the subspace $V_1$ that satisfies the hypotheses of Lemma 7 from Chapter III.

Let us now compute the centralizer of $V_1$. By Lemma 14 in Chapter II, each element of $C(V_1)$ has form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & 0 \end{pmatrix}.$$

Note that $x \in C(V_1)$ if and only if $\partial_1(x) \in V_1$ and $\partial_2(x) \in V_1$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & 0 \\ x_{1,1} & x_{1,2} & 0 \\ x_{2,1} & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_1$ if and only if

$$x_{1,2} = c_1 \cdot x_{2,1},$$

and that $\partial_2(x) \in V_1$ if and only if

$$x_{2,1} = 0.$$

Hence, $x \in C(V_1)$ if and only if

$$x_{2,1} = 0$$

$$x_{1,2} = c_1 \cdot x_{2,1} = 0.$$

Hence, $C(V_1) = C(V_0)$. Thus, $\dim(C(V_1)) - \dim(V_1) = 2$.

Case 2: Fix $c_1, c_2 \in \mathbb{Z}_3$ such that $(c_1, c_2) \neq (0,0)$ and define

$$m_1 = \begin{pmatrix} 0 & 0 & c_2 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$B_1 = E(\alpha_2) \cup \{m_1\}$, and $V_1 = \langle B_1 \rangle$. Note that $B_1$ is a basis for the subspace $V_1$ that satisfies the hypotheses of Lemma 7 from Chapter III.

Let us now compute the centralizer of $V_1$. By Lemma 14 in Chapter II, each element of $C(V_1)$ has form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & 0 \end{pmatrix}.$$

Note that $x \in C(V_1)$ if and only if $\partial_1(x) \in V_1$ and $\partial_2(x) \in V_1$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & 0 \\ x_{1,1} & x_{1,2} & 0 \\ x_{2,1} & 0 & 0 \end{pmatrix}.$$

Suppose $\partial_1(x) \in V_1$, and consider the expansion of $\partial_1(x)$ with respect to the basis $B_1$. By Lemma 7 in Chapter III the coefficient on $m_1$ in the expansion of $\partial_1(x)$ must be 0. This would imply that $x_{2,1} = x_{1,2} = 0$ and $\dim(C(V_1)) - \dim(V_1) = 2$.

Hence, in either case, we want the 2-dimensional good subspaces of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. From the construction of such spaces in Chapter III, we know that there are four. These are generated by sets of vectors of the form $\{(1, 0, c_3), (0, 1, c_4)\}$ where $c_3, c_4 \neq 0$. Thus, fix non-zero scalars $c_3, c_4 \in \mathbb{Z}_3$ and define

$$m_2 = \begin{pmatrix} 0 & 0 & c_3 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad m_3 = \begin{pmatrix} 0 & 0 & c_4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$B_2 = E(\alpha_2) \cup \{m_2, m_3\}$, and $V_2 = \langle B_2 \rangle$. Note that $B_2$ is a basis for the subspace $V_2$ that satisfies the hypotheses of Lemma 7 in Chapter III.

Let us now compute the centralizer of $V_2$. By Lemma 14 in Chapter II, each element of $C(V_2)$ has form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & 0 \end{pmatrix}.$$

Note that $x \in C(V_2)$ if and only if $\partial_1(x) \in V_2$ and $\partial_2(x) \in V_2$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & 0 \\ x_{1,1} & x_{1,2} & 0 \\ x_{2,1} & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_2$ if and only if

$$x_{1,2} = c_4 \cdot x_{2,1},$$

and that $\partial_2(x) \in V_2$ if and only if

$$c_3 \cdot x_{2,1} + c_4 \cdot x_{1,2} = 0.$$

Hence, $x \in C(V_2)$ if and only if

$$c_3 \cdot x_{2,1} + c_4^2 \cdot x_{2,1} = (c_3 + c_4^2) \cdot x_{2,1} = 0.$$

Hence, there are two cases. In the first case, $c_3 \neq -c_4^2$. This implies that $x_{2,1} = 0$ since the entries in the matrices come from a field. This further shows that $x_{1,2} = 0$. Thus, $C(V_2) = C(V_1) = C(V_0)$ which implies that $\dim(C(V_2)) - \dim(V_2) = 1$ and the algorithm terminates. However, in the second case, if $c_3 = -c_4^2$, then $\dim(C(V_2)) - \dim(V_2) = 2$. More explicitly, if we define

$$v_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_4 \\ 0 & 1 & 0 \end{pmatrix},$$

and $B_2' = B_0' \cup \{v_1\}$, then $C(V_2) = \langle B_2' \rangle$. Since $\dim(C(V_2)) - \dim(V_2) = 2$, we must choose a matrix $m_4 \in C(V_2) - V_2$ such that for each point $(i, j) \in \mathcal{U}$, if $e_{i,j} \in \langle V_2, m_1 \rangle$, then $e_{i,j} \in V_2$. If the (2,0)-entry of $m_4$ is non-zero, we can subtract off appropriate multiples of $m_4$ to obtain a new matrix whose (2,0)-entry is 0. We may do this since the (2,0)-entry of $m_2$ is 1. Hence, we assume without loss of generality that the (2,0)-entry of $m_4$ is 0. For similar reasons, we assume without loss of generality that the (1,1)-entry of $m_4$ is 0. Consider the expansion of $m_4$ with respect to the basis $B_2'$. If the coefficient on $v_1$ in the expansion of $m_4$ is 0, then the only non-zero entry of $m_4$

is the (0,2)-entry. However, that would mean $e_{0,2} \in \langle V_1, m_4 \rangle$ even though $e_{0,2} \notin V_1$. Thus, the coefficient on $v_1$ in the expansion of $m_4$ must be non-zero. Since we may scale the coefficient on $v_1$, we shall choose said coefficient to be 1. Hence, fix $c_5 \in \mathbb{Z}_3$ and define the matrix

$$m_4 = \begin{pmatrix} 0 & 0 & c_5 \\ 0 & 0 & c_4 \\ 0 & 1 & 0 \end{pmatrix},$$

$B_3 = B_2 \cup \{m_4\}$, and $V_3 = \langle B_3 \rangle$. Note that $B_3$ is a basis for the subspace $V_3$ that satisfies the hypotheses of Lemma 7 in Chapter III. Lemma 14 in Chapter II demonstrates that the elements in $C(V_3)$ will take the form shown below.

Let us now compute the centralizer of $V_3$. Each element of $C(V_3)$ has form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & x_{2,2} \end{pmatrix}.$$

Note that $x \in C(V_3)$ if and only if $\partial_1(x) \in V_3$ and $\partial_2(x) \in V_3$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & x_{2,2} \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & 0 \\ x_{1,1} & x_{1,2} & 0 \\ x_{2,1} & x_{2,2} & 0 \end{pmatrix}.$$

Suppose $\partial_1(x) \in V_3$, and consider the expansion of $\partial_1(x)$ with respect to the basis $B_3$. By Lemma 7 in Chapter III the coefficient on $m_4$ in the expansion of $\partial_1(x)$ must be 0. This would imply that $x_{2,2} = 0$ and $\dim(C(V_3)) - \dim(V_3) = 1$ and the algorithm terminates.

Table 5.2: Results For The Pattern $\alpha_2$

| $\text{ht}(\alpha_2)$ | $\lvert\mathcal{V}(\alpha_2)_0\rvert$ | $\lvert\mathcal{V}(\alpha_2)_1\rvert$ | $\lvert\mathcal{V}(\alpha_2)_2\rvert$ | $\lvert\mathcal{V}(\alpha_2)_3\rvert$ | $\lvert\mathcal{V}(\alpha_2)\rvert$ |
|---|---|---|---|---|---|
| 3 | 1 | 10 | 4 | 2 | 17 |

This is the first of many investigations in which our arguments must be broken up on the basis of what we call coefficient relationships. This will cause added difficulty in presenting a summary. Fortunately, the coefficient relationship $c_3 = -c_4^2$ is sufficiently simple. By inspection, the only pairs of numbers $(c_3, c_4)$ that satisfy the equation $c_3 = -c_4^2$ are $c_3 = 2, c_4 = 1$ and $c_3 = c_4 = 2$. We summarize the result of this pattern in Table 5.2 .

The other patterns are handled in a manner similar to all of the patterns for the case $p = 2$. For instance, consider the pattern

$$\alpha = \begin{pmatrix} \bullet & 0 & 0 \\ \bullet & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We want to list the subspaces in the set $\mathcal{V}(\alpha)$. To this end define the basis $B_0 = E(\alpha)$ and the subspace $V_0 = V(\alpha) = \langle B_0 \rangle$. Theorem 16 from Chapter II demonstrates that $C(V_0) = V(\beta)$ where

$$\beta = \begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & 0 & 0 \\ \bullet & 0 & 0 \end{pmatrix}.$$

Write $B_0' = E(\beta)$ and note $V(\beta) = \langle B_0' \rangle$. Since $\dim(C(V_0)) - \dim(V_0) = 2$, we must choose a matrix $m \in C(V_0) - V_0$ such that for each point $(i, j) \in \mathcal{U}$ if $e_{i,j} \in \langle V_0, m \rangle$, then $e_{i,j} \in V_0$. Hence, we want the 1-dimensional good subspaces of $\mathbb{Z}_3 \times \mathbb{Z}_3$. From the construction of such spaces in Chapter III, we know that these are generated by the vectors of the form $(1, c)$ where $c \neq 0$. To this end, fix a non-zero scalar $c \in \mathbb{Z}_3$ and define

$$m = \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$B_1 = B_0 \cup \{m\}$, and $V_1 = \langle B_1 \rangle$. Note that $B_1$ is a basis of the subspace $V_1$ that satisfies the hypotheses of Lemma 7 from Chapter III.

Let us now compute the centralizer of $V_1$. By Lemma 14 in Chapter II, each element of $C(V_1)$ has form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & 0 \\ x_{2,0} & x_{2,1} & 0 \end{pmatrix}.$$

Note that $x \in C(V_1)$ if and only if $\partial_1(x) \in V_1$ and $\partial_2(x) \in V_1$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & 0 \\ x_{2,0} & x_{2,1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & 0 \\ x_{1,1} & 0 & 0 \\ x_{2,1} & 0 & 0 \end{pmatrix}.$$

Suppose $\partial_1(x) \in V_1$, and consider the expansion of $\partial_1(x)$ with respect to the basis $B_1$. By Lemma 7 in Chapter III the coefficient on $m$ in the expansion of $\partial_1(x)$ must

Table 5.3: Results For The Pattern $\alpha$

| $\mathrm{ht}(\alpha)$ | $\lvert\mathcal{V}(\alpha)_0\rvert$ | $\lvert\mathcal{V}(\alpha)_1\rvert$ | $\lvert\mathcal{V}(\alpha)\rvert$ |
|---|---|---|---|
| 1 | 1 | 2 | 3 |

be 0. This would imply that $x_{1,1} = 0$. Also, $\partial_1(x) \in V_1$ implies that $x_{2,1} = 0$. Hence, $C(V_1) = C(V_0)$ and $\dim(C(V_1)) - \dim(V_1) = 1$. Thus our algorithm terminates and we are finished with this pattern. A summary is given in Table 5.3.

Computing $\mathcal{V}(\alpha)$ is handled in a similar manner for the patterns below (the reader is invited to verify this). Using the language we used in Chapter II to motivate Theorem 16, the reader should note that each of these patterns has exactly two "corners":

$$\begin{pmatrix} \bullet & \bullet & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & 0 & 0 \\ \bullet & 0 & 0 \end{pmatrix}, \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & \bullet & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & \bullet & 0 \\ \bullet & 0 & 0 \end{pmatrix}, \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{and,} \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & 0 \\ \bullet & 0 & 0 \end{pmatrix}.$$

For the above patterns, it is the case that $|\mathcal{V}(\alpha)| = 3$. For any other pattern $\gamma \in \mathcal{P}$ it is the case that $\dim(C(V(\gamma))) - \dim(V(\gamma)) \leq 1$. The reader can verify this by applying Theorem 16 to any pattern in question. However, there is an easier way to see this using the language we used in Chapter II to motivate Theorem 16.

The reader should note that each of these patterns has at most one "corner". These patterns are:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \bullet & 0 & 0 \\ \bullet & 0 & 0 \\ \bullet & 0 & 0 \end{pmatrix}, \begin{pmatrix} \bullet & \bullet & \bullet \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & 0 & 0 \\ \bullet & 0 & 0 \end{pmatrix}, \begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & \bullet & 0 \\ \bullet & \bullet & 0 \end{pmatrix},$$

$$\begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & 0 \\ \bullet & \bullet & 0 \end{pmatrix}, \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & 0 & 0 \end{pmatrix}, \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & 0 \end{pmatrix} \text{and,} \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}.$$

Thus, when $p = 3$, there are 60 doubly-invariant subspaces of $\mathcal{M}$. That concludes the results for the case $p = 3$.

# CHAPTER VI

# THE RESULTS FOR THE PRIME 5

The total results for the case when $p = 5$ are not even close to being completed in this thesis. The case $p = 5$ is a much more difficult and computationally intensive case than when our prime was 2 or 3. However, this means that it is also much more interesting. In particular, we will start to see some behavior which gives us an idea about when the centralizer of a subspace will get larger and when it will not. While such ideas are not yet formalized, they are mentioned as a starting point for further research beyond this thesis.

When our prime was 2, the only interesting pattern was

$$\begin{pmatrix} \bullet & 0 \\ 0 & 0 \end{pmatrix}.$$

When our prime was 3, the only interesting patterns were

$$\begin{pmatrix} \bullet & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The common theme among these patterns is that they contain "waves" of dots filling up the "anti-diagonals" up to but not including the main anti-diagonal. Since $p = 5$ has been much too large to do completely, we need to focus on a few particularly interesting patterns. Following the above examples, it seems like the patterns to focus on are the following:

$$\begin{pmatrix} \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \bullet & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and, } \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This chapter is long and only the first of the above patterns has been investigated completely. Because of this, the content of this chapter has been broken into sections on the basis of the pattern being considered. Each section has an introductory paragraph which gives the reader an idea of how the results are organized.

## 6.1 The First Wave Pattern

This pattern is the only straightforward pattern in Chapter VI. It does not contain multiple cases which must be considered separately. We shall simply iterate our algorithm for computing doubly-invariant spaces four times.

We begin with the pattern

$$\alpha_1 = \begin{pmatrix} \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We want to list the subspaces in the set $\mathcal{V}(\alpha_1)$. To this end write $B_0 = E(\alpha_1)$ and $V_0 = V(\alpha_1) = \langle B_0 \rangle$. Theorem 16 from Chapter II demonstrates that $C(V_0) = V(\beta_1)$ where

$$\beta_1 = \begin{pmatrix} \bullet & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Write $B_0' = E(\beta_1)$ and note that $V(\beta_1) = \langle B_0' \rangle$. Since $\dim(C(V_0)) - \dim(V_0) = 2$, we must choose a matrix $m_1 \in C(V_0) - V_0$ such that for each point $(i, j) \in \mathcal{U}$, if $e_{i,j} \in \langle V_0, m_1 \rangle$, then $e_{i,j} \in V_0$. Hence, we want the 1-dimensional good subspaces of $\mathbb{Z}_5 \times \mathbb{Z}_5$. From the construction of such spaces in Chapter III, we know that such subspaces are generated by vectors of the form $(1, c_1)$ where $c_1 \neq 0$. Fix a non-zero scalar $c_1 \in \mathbb{Z}_5$ and define

$$m_1 = \begin{pmatrix} 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$B_1 = B_0 \cup \{m_1\}$, and $V_1 = \langle B_1 \rangle$. Note that $B_1$ is a basis for the subspace $V_1$ that satisfies the hypotheses of Lemma 7 in Chapter III.

Let us now compute the centralizer of $V_1$. By Lemma 14 in Chapter II, each element of $C(V_1)$ has form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & 0 & 0 \\ x_{1,0} & x_{1,1} & 0 & 0 & 0 \\ x_{2,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_1)$ if and only if $\partial_1(x) \in V_1$ and $\partial_2(x) \in V_1$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & 0 & 0 & 0 \\ x_{2,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & 0 & 0 & 0 \\ x_{1,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_1$ if and only if

$$x_{1,1} = c_1 \cdot x_{2,0}$$

and that $\partial_2(x) \in V_1$ if and only if

$$x_{0,2} = c_1 \cdot x_{1,1}.$$

Hence, $x \in C(V_1)$ if and only if

$$x_{1,1} = c_1 \cdot x_{2,0}$$

$$x_{0,2} = c_1 \cdot x_{1,1} = c_1^2 \cdot x_{2,0}.$$

Therefore, we define

$$v_1 = \begin{pmatrix} 0 & 0 & c_1^2 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $B_1' = B_0' \cup \{v_1\}$ and note that $C(V_1) = \langle B_1' \rangle$. Since $\dim(C(V_1)) - \dim(V_1) = 2$, we must choose a matrix $m_2 \in C(V_1) - V_1$ such that for each point $(i, j) \in \mathcal{U}$, if $e_{i,j} \in \langle V_1, m_2 \rangle$, then $e_{i,j} \in V_1$. If the (1,0)-entry of $m_2$ is non-zero, we can subtract off appropriate multiples of $m_1$ to obtain a new matrix whose (1,0)-entry is 0. We may do this since the (1,0)-entry of $m_1$ is 1. Hence, we may assume without loss of generality that the (1,0)-entry of $m_2$ is 0. Consider the expansion of $m_2$ with respect to the basis $B_1'$. If the coefficient on $v_1$ in the expansion of $m_2$ is 0, then the only non-zero entry of $m_2$ is the (0,1)-entry. However, that would mean $e_{0,1} \in \langle V_1, m_2 \rangle$ even though $e_{0,1} \notin V_1$. Thus, the coefficient on $v_1$ in the expansion of $m_2$ must be non-zero. Since we may scale the coefficient on $v_1$, we shall choose said coefficient to be 1. Hence, we fix $c_2 \in \mathbb{Z}_5$ and define the matrix

$$m_2 = \begin{pmatrix} 0 & c_2 & c_1^2 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$B_2 = B_1 \cup \{m_2\}$, and $V_2 = \langle B_2 \rangle$. Note that we may have $c_2 = 0$ without worrying about having a standard basis vector in our span. Note that $B_2$ is a basis for the subspace $V_2$ that satisfies the hypotheses of Lemma 7 in Chapter III.

Let us now compute the centralizer of $V_2$. By Lemma 14 in Chapter II, each element of $C(V_2)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & 0 \\ x_{1,0} & x_{1,1} & x_{1,2} & 0 & 0 \\ x_{2,0} & x_{2,1} & 0 & 0 & 0 \\ x_{3,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_2)$ if and only if $\partial_1(x) \in V_2$ and $\partial_2(x) \in V_2$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & 0 & 0 \\ x_{2,0} & x_{2,1} & 0 & 0 & 0 \\ x_{3,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & 0 & 0 \\ x_{1,1} & x_{1,2} & 0 & 0 & 0 \\ x_{2,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_2$ if and only if

$$x_{2,1} = c_1 \cdot x_{3,0}$$

$$x_{1,2} = c_1^2 \cdot x_{3,0}$$

$$x_{1,1} = c_1 \cdot x_{2,0} + c_2 \cdot x_{3,0}$$

and that $\partial_2(x) \in V_2$ if and only if

$$x_{1,2} = c_1 \cdot x_{2,1}$$

$$x_{0,3} = c_1^2 \cdot x_{2,1}$$

$$x_{0,2} = c_1 \cdot x_{1,1} + c_2 \cdot x_{2,1}.$$

Hence, $x \in C(V_2)$ if and only if

$$x_{1,1} = c_1 \cdot x_{2,0} + c_2 \cdot x_{3,0}$$

$$x_{2,1} = c_1 \cdot x_{3,0}$$

$$x_{1,2} = c_1^2 \cdot x_{3,0}$$

$$x_{0,3} = c_1^2 \cdot x_{2,1} = c_1^3 \cdot x_{3,0}$$

$$x_{0,2} = c_1 \cdot x_{1,1} + c_2 \cdot x_{2,1} = c_1^2 \cdot x_{2,0} + c_1 c_2 \cdot x_{3,0} + c_1 c_2 \cdot x_{3,0} = c_1^2 \cdot x_{2,0} + 2c_1 c_2 \cdot x_{3,0}.$$

Therefore, we define

$$v_2 = \begin{pmatrix} 0 & 0 & 2c_1c_2 & c_1^3 & 0 \\ 0 & c_2 & c_1^2 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $B_2' = B_1' \cup \{v_2\}$ and note that $C(V_2) = \langle B_2' \rangle$. Since $\dim(C(V_2)) - \dim(V_2) = 2$, we must choose a matrix $m_3 \in C(V_2) - V_2$ such that for each $(i, j) \in \mathcal{U}$, if $e_{i,j} \in \langle V_2, m_3 \rangle$, then $e_{i,j} \in V_2$. Similar to how we argued above, we may assume without loss of generality that the (1,0)-entry and the (2,0)-entry of $m_3$ are 0. Consider the expansion of $m_3$ with respect to the basis $B_2'$. Note that the coefficient on $v_1$ in the expansion of $m_3$ is 0 since we have assumed that the (2,0)-entry of $m_3$ is 0. Hence, if the coefficient on $v_2$ in the expansion of $m_3$ is 0, then this would mean that $e_{0,1} \in \langle V_2, m_3 \rangle$ even though $e_{0,1} \notin V_2$. Thus, the coefficient on $v_2$ in the expansion of $m_3$ must be non-zero. Since we may scale the coefficient on $v_2$, we shall choose said coefficient to be 1. Hence, we fix $c_3 \in \mathbb{Z}_5$ and define the matrix

$$m_3 = \begin{pmatrix} 0 & c_3 & 2c_1c_2 & c_1^3 & 0 \\ 0 & c_2 & c_1^2 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$B_3 = B_2 \cup \{m_3\}$, and $V_3 = \langle B_3 \rangle$. Note that we may have $c_3 = 0$ without worrying about having a standard basis vector in our span. Note that $B_3$ is a basis for the subspace $V_3$ that satisfies the hypotheses of Lemma 7 in Chapter III.

Let us now compute the centralizer of $V_3$. By Lemma 14 in Chapter II, each element of $C(V_3)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_3)$ if and only if $\partial_1(x) \in V_3$ and $\partial_2(x) \in V_3$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & 0 & 0 \\ x_{2,1} & x_{2,2} & 0 & 0 & 0 \\ x_{3,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_3$ if and only if

$$x_{3,1} = c_1 \cdot x_{4,0}$$

$$x_{2,2} = c_1^2 \cdot x_{4,0}$$

$$x_{1,3} = c_1^3 \cdot x_{4,0}$$

$$x_{2,1} = c_1 \cdot x_{3,0} + c_2 \cdot x_{4,0}$$

$$x_{1,2} = c_1^2 \cdot x_{3,0} + 2c_1c_2 \cdot x_{4,0}$$

$$x_{1,1} = c_1 \cdot x_{2,0} + c_2 \cdot x_{3,0} + c_3 \cdot x_{4,0}$$

and that $\partial_2(x) \in V_3$ if and only if

$$x_{2,2} = c_1 \cdot x_{3,1}$$

$$x_{1,3} = c_1^2 \cdot x_{3,1}$$

$$x_{0,4} = c_1^3 \cdot x_{3,1}$$

$$x_{1,2} = c_1 \cdot x_{2,1} + c_2 \cdot x_{3,1}$$

$$x_{0,3} = c_1^2 \cdot x_{2,1} + 2c_1c_2 \cdot x_{3,1}$$

$$x_{0,2} = c_1 \cdot x_{1,1} + c_2 \cdot x_{2,1} + c_3 \cdot x_{3,1}.$$

Hence, $x \in C(V_3)$ if and only if

$$
\begin{aligned}
x_{3,1} &= c_1 \cdot x_{4,0} \\
x_{2,2} &= c_1^2 \cdot x_{4,0} \\
x_{1,3} &= c_1^3 \cdot x_{4,0} \\
x_{0,4} &= c_1^3 \cdot x_{3,1} = c_1^4 \cdot x_{4,0} \\
x_{1,2} &= c_1 \cdot x_{2,1} + c_2 \cdot x_{3,1} = c_1^2 \cdot x_{3,0} + c_1c_2 \cdot x_{4,0} + c_1c_2 \cdot x_{4,0} = c_1^2 \cdot x_{3,0} + 2c_1c_2 \cdot x_{4,0} \\
x_{0,3} &= c_1^2 \cdot x_{2,1} + 2c_1c_2 \cdot x_{3,1} = c_1^2(c_1 \cdot x_{3,0} + c_2 \cdot x_{4,0}) + 2c_1c_2(c_1 \cdot x_{4,0}) \\
&= c_1^3 \cdot x_{3,0} + 3c_1^2c_2 \cdot x_{4,0} \\
x_{0,2} &= c_1 \cdot x_{1,1} + c_2 \cdot x_{2,1} + c_3 \cdot x_{3,1} \\
&= c_1(c_1 \cdot x_{2,0} + c_2 \cdot x_{3,0} + c_3 \cdot x_{4,0}) + c_2(c_1 \cdot x_{3,0} + c_2 \cdot x_{4,0}) + c_3(c_1 \cdot x_{4,0}) \\
&= c_1^2 \cdot x_{2,0} + 2c_1c_2 \cdot x_{3,0} + (2c_1c_3 + c_2^2) \cdot x_{4,0}.
\end{aligned}
$$

Therefore, we define

$$
v_3 = \begin{pmatrix} 0 & 0 & 2c_1c_3 + c_2^2 & 3c_1^2c_2 & c_1^4 \\ 0 & c_3 & 2c_1c_2 & c_1^3 & 0 \\ 0 & c_2 & c_1^2 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

and $B_3' = B_2' \cup \{v_3\}$ and note that $C(V_3) = \langle B_3' \rangle$. Since $\dim(C(V_3)) - \dim(V_3) = 2$, we must choose a matrix $m_4 \in C(V_3) - V_3$ such that for each $(i,j) \in \mathcal{U}$, if $e_{i,j} \in \langle V_3, m_4 \rangle$, then $e_{i,j} \in V_3$. Similar to how we argued above, we assume without loss of generality that the (1,0)-entry, the (2,0)-entry, and the (3,0)-entry of $m_4$ are 0. Consider the expansion of $m_4$ with respect to the basis $B'_3$. Note that the coefficient on $v_1$ in the expansion of $m_4$ is 0 since we have assumed that the (2,0)-entry of $m_4$ is 0. Similarly, the coefficient on $v_2$ in the expansion of $m_4$ is also 0. Hence, if the coefficient on $v_3$ in the expansion of $m_4$ is 0, then this would mean that $e_{0,1} \in \langle V_3, m_4 \rangle$ even though $e_{0,1} \notin V_3$. Thus, the coefficient on $v_3$ in the expansion of $m_4$ must be non-zero. Since we may scale the coefficient on $v_3$, we shall choose said coefficient to be 1. Hence, we fix $c_4 \in \mathbb{Z}_5$ and define the matrix

$$m_4 = \begin{pmatrix} 0 & c_4 & 2c_1c_3 + c_2^2 & 3c_1^2c_2 & c_1^4 \\ 0 & c_3 & 2c_1c_2 & c_1^3 & 0 \\ 0 & c_2 & c_1^2 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$B_4 = B_3 \cup \{m_4\}$, and $V_4 = \langle B_4 \rangle$. Note that we may have $c_4 = 0$ without worrying about having a standard basis vector in our span. Note that $B_4$ is a basis for the subspace $V_4$ that satisfies the hypotheses of Lemma 7 in Chapter III.

Let us now compute the centralizer of $V_4$. By Lemma 14 in Chapter II, each

element of $C(V_4)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_4)$ if and only if $\partial_1(x) \in V_4$ and $\partial_2(x) \in V_4$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Suppose $\partial_1(x) \in V_4$, and consider the expansion of $\partial_1(x)$ with respect to the basis $B_4$. By Lemma 7 in Chapter III the coefficient on $m_4$ in the expansion of $\partial_1(x)$ must be 0. This would imply that $x_{4,1} = x_{3,2} = x_{2,3} = x_{1,4} = 0$. Further, the representations of the other entries do not change since the coefficient on $m_4$ is 0 in the expansions of both $\partial_1(x)$ and $\partial_2(x)$. Hence, $C(V_4) = C(V_3)$, which implies that $\dim(C(V_4)) - \dim(V_4) = 1$ and the algorithm terminates. The results are summarized in table 6.1.

Note that with each new matrix we added to our basis, our centralizers grow. More precisely, for each $i \in \{0, 1, 2\}$ it is the case that $\dim(C(V_i)) < \dim(C(V_{i+1}))$. Also, note that each $m_i$ has leading non-zero anti-diagonal has entries which form a

Table 6.1: Results For The Pattern $\alpha_1$

| $\text{ht}(\alpha_1)$ | $\lvert\mathcal{V}(\alpha_1)_0\rvert$ | $\lvert\mathcal{V}(\alpha_1)_1\rvert$ | $\lvert\mathcal{V}(\alpha_1)_2\rvert$ | $\lvert\mathcal{V}(\alpha_1)_3\rvert$ | $\lvert\mathcal{V}(\alpha_1)_4\rvert$ | $\lvert\mathcal{V}(\alpha_1)\rvert$ |
|---|---|---|---|---|---|---|
| 4 | 1 | 4 | 20 | 100 | 500 | 625 |

geometric sequence, since the first entry is 1, the second is $c_1$, and the third is $c_1^2$. In general, the $n$th entry was $c_1^{n-1}$. This theme will repeat itself.

The reader should recall that if some subspace $V_i \in \mathcal{V}$ satisfies the condition $\dim(C(V_i)) - \dim(V_i) \geq 2$, then we choose some matrix $m \in C(V_i) - V_i$ and consider the subspace $V_{i+1} = \langle V_i, m\rangle$. Up to this point we have argued somewhat extensively for the particular nature of the matrix $m$. Since these arguments have now been made numerous times and are not difficult to understand, said arguments will be omitted in future investigations and $m$ will be defined with little comment.

## 6.2 The Second Wave Pattern

This pattern will not be fully explored. After one iteration of our algorithm, we will define Case 1 and Case 2 since there are two different ways to define the matrix $m_1$. Case 1 will be further subdivided into Case 1.1 and Case 1.2 on the basis of a possible relationship between the entries of the matrix $m_1$. After one iteration of our algorithm for Cases 1.2 and 2, we will be able to treat them simultaneously. Hence, we will combine them into a single case, which we will call Case 3. Case 3 will then be subdivided into Case 3.1 and Case 3.2 since there will be two different ways to define the matrix $m_4$. After one iteration of our algorithm for Case 3.1 and Case 3.2 we will cease investigating them. We will then return to Case 1.1 and perform one more iteration of our algorithm for it.

We define the pattern

$$\alpha_2 = \begin{pmatrix} \bullet & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and write $V_0 = V(\alpha_2)$. Clearly, $C(V_0) = V(\beta_2)$ where

$$\beta_2 = \begin{pmatrix} \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we want the one-dimensional good subspaces of $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$. There are two types of vectors which produce this. The first is $(1, c_1, c_2)$ where $(c_1, c_2) \neq (0, 0)$. The second has the form $(0, 1, c_1)$ where $c_1 \neq 0$. Hence, we break our investigation into two cases.

Case 1: Fix $c_1, c_2 \in \mathbb{Z}_5$ such that $(c_1, c_2) \neq (0,0)$ and define

$$m_1 = \begin{pmatrix} 0 & 0 & c_2 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_1 = \langle V_0, m_1 \rangle$.

Let us now compute the centralizer of $V_1$. By Lemma 14 in Chapter II, each element of $C(V_1)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & 0 \\ x_{1,0} & x_{1,1} & x_{1,2} & 0 & 0 \\ x_{2,0} & x_{2,1} & 0 & 0 & 0 \\ x_{3,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_1)$ if and only if $\partial_1(x) \in V_1$ and $\partial_2(x) \in V_1$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & 0 & 0 \\ x_{2,0} & x_{2,1} & 0 & 0 & 0 \\ x_{3,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & 0 & 0 \\ x_{1,1} & x_{1,2} & 0 & 0 & 0 \\ x_{2,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_1$ if and only if

$$x_{2,1} = c_1 \cdot x_{3,0}$$

$$x_{1,2} = c_2 \cdot x_{3,0},$$

and that $\partial_2(x) \in V_1$ if and only if

$$x_{1,2} = c_1 \cdot x_{2,1}$$

$$x_{0,3} = c_2 \cdot x_{2,1}.$$

Hence, $x \in C(V_1)$ if and only if

$$x_{2,1} = c_1 \cdot x_{3,0}$$

$$x_{1,2} = c_2 \cdot x_{3,0} = c_1 \cdot x_{2,1} = c_1^2 \cdot x_{3,0}$$

$$x_{0,3} = c_2 \cdot x_{2,1} = c_1 c_2 \cdot x_{3,0}.$$

Note that the second equation implies that $(c_1^2 - c_2) \cdot x_{3,0} = 0$. Thus, we have two subcases: either $c_1^2 = c_2$ or $c_1^2 \neq c_2$.

Case 1.1: Suppose that $c_1^2 = c_2$. Hence we have

$$x_{2,1} = c_1 \cdot x_{3,0}$$

$$x_{1,2} = c_2 \cdot x_{3,0} = c_1 \cdot x_{2,1} = c_1^2 \cdot x_{3,0}$$

$$x_{0,3} = c_2 \cdot x_{2,1} = c_1 c_2 \cdot x_{3,0} = c_1^3 \cdot x_{3,0}.$$

Thus, we define

$$v_1 = \begin{pmatrix} 0 & 0 & 0 & c_1^3 & 0 \\ 0 & 0 & c_1^2 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and note that $C(V_1) = \langle V_0, v_1 \rangle$. We shall return to this case later.

Case 1.2: Suppose $c_1^2 \neq c_2$. Then we must have $x_{3,0} = 0$. This implies that $x_{2,1} = x_{1,2} = x_{0,3} = 0$. Hence, if $x \in C(V_1)$, then $x \in C(V_0)$. This yields that $C(V_1) = C(V_0)$, and so $\dim(C(V_1)) - \dim(V_1) = 2$. Therefore, we are interested in the two-dimensional good subspaces of $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ and will compute them later.

Case 2: Fix a non-zero scalar $c_1 \in \mathbb{Z}_5$ and define

$$m_1 = \begin{pmatrix} 0 & 0 & c_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_1 = \langle V_0, m_1 \rangle$.

Let us now compute the centralizer of $V_1$. By Lemma 14 in Chapter II, each

element of $C(V_1)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & 0 \\ x_{1,0} & x_{1,1} & x_{1,2} & 0 & 0 \\ x_{2,0} & x_{2,1} & 0 & 0 & 0 \\ x_{3,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_1)$ if and only if $\partial_1(x) \in V_1$ and $\partial_2(x) \in V_1$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & 0 & 0 \\ x_{2,0} & x_{2,1} & 0 & 0 & 0 \\ x_{3,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & 0 & 0 \\ x_{1,1} & x_{1,2} & 0 & 0 & 0 \\ x_{2,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_1$ if and only if

$$x_{3,0} = 0$$

$$x_{1,2} = c_1 \cdot x_{2,1},$$

and that $\partial_2(x) \in V_1$ if and only if

$$x_{2,1} = 0$$

$$x_{0,3} = c_1 \cdot x_{1,2}.$$

Hence, $x \in C(V_1)$ if and only if

$$x_{3,0} = 0$$

$$x_{2,1} = 0$$

$$x_{1,2} = c_1 \cdot x_{2,1} = 0$$

$$x_{0,3} = c_1 \cdot x_{1,2} = 0.$$

Hence, $C(V_1) = C(V_0)$ which yields that $\dim(C(V_1)) - \dim(V_1) = 2$. Therefore, we are interested in the two-dimensional good subspaces of $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$.

Note that Case 1.2 and Case 2 ended in the same conclusion. We shall combine these into a single case, which we call Case 3.

Case 3: We want the two-dimensional good subspaces of $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$. These are spanned by a set of two vectors which have the forms $(1, 0, c_3)$ and $(0, 1, c_4)$ where $c_3, c_4 \neq 0$. To this end fix non-zero scalars $c_3, c_4 \in \mathbb{Z}_5$ and define

$$m_2 = \begin{pmatrix} 0 & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad m_3 = \begin{pmatrix} 0 & 0 & c_4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $V_2 = \langle V_0, m_2, m_3 \rangle$.

We shall now compute the centralizer of $V_2$. By Lemma 14 in Chapter II,

each element of $C(V_2)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & 0 \\ x_{1,0} & x_{1,1} & x_{1,2} & 0 & 0 \\ x_{2,0} & x_{2,1} & 0 & 0 & 0 \\ x_{3,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_2)$ if and only if $\partial_1(x) \in V_2$ and $\partial_2(x) \in V_2$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & 0 & 0 \\ x_{2,0} & x_{2,1} & 0 & 0 & 0 \\ x_{3,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & 0 & 0 \\ x_{1,1} & x_{1,2} & 0 & 0 & 0 \\ x_{2,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_2$ if and only if

$$x_{1,2} = c_3 \cdot x_{3,0} + c_4 \cdot x_{2,1},$$

and that $\partial_2(x) \in V_2$ if and only if

$$x_{0,3} = c_3 \cdot x_{2,1} + c_4 \cdot x_{1,2}.$$

Hence, $x \in C(V_2)$ if and only if

$$x_{1,2} = c_3 \cdot x_{3,0} + c_4 \cdot x_{2,1}$$

$$x_{0,3} = c_3 \cdot x_{2,1} + c_4 \cdot x_{1,2} = c_3 c_4 \cdot x_{3,0} + (c_3 + c_4^2) \cdot x_{2,1}.$$

Hence, we define

$$v_1 = \begin{pmatrix} 0 & 0 & 0 & c_3c_4 & 0 \\ 0 & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad v_2 = \begin{pmatrix} 0 & 0 & 0 & c_3 + c_4^2 & 0 \\ 0 & 0 & c_4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that $C(V_2) = \langle V(\beta_2), v_1, v_2 \rangle$. Hence, $\dim(C(V_2)) - \dim(V_2) = 3$. Since there are two matrices in our centralizer which dictate the entries on the fourth antidiagonal, we have two choices regarding the construction of our next matrix.

Case 3.1: Fix $c_5 \in \mathbb{Z}_5$ and define

$$m_4 = \begin{pmatrix} 0 & 0 & c_5 & c_3 + c_4^2 & 0 \\ 0 & 0 & c_4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_3 = \langle V_0, m_2, m_3, m_4 \rangle$.

We shall now compute the centralizer of $V_3$. By Lemma 14 in Chapter II,

each element of $C(V_3)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_3)$ if and only if $\partial_1(x) \in V_3$ and $\partial_2(x) \in V_3$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & 0 & 0 \\ x_{2,1} & x_{2,2} & 0 & 0 & 0 \\ x_{3,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_3$ if and only if

$$x_{1,2} = c_3 \cdot x_{3,0} + c_4 \cdot x_{2,1} + c_5 \cdot x_{3,1}$$

$$x_{4,0} = 0$$

$$x_{2,2} = c_4 \cdot x_{3,1}$$

$$x_{1,3} = (c_3 + c_4^2) \cdot x_{3,1},$$

and that $\partial_2(x) \in V_3$ if and only if

$$x_{0,3} = c_3 \cdot x_{2,1} + c_4 \cdot x_{1,2} + c_5 \cdot x_{3,1}$$

$$x_{3,1} = 0$$

$$x_{1,3} = c_4 \cdot x_{2,2}$$

$$x_{0,4} = (c_3 + c_4^2) \cdot x_{2,2}.$$

Hence, $x \in C(V_3)$ if and only if

$$x_{4,0} = 0$$

$$x_{3,1} = 0$$

$$x_{2,2} = c_4 \cdot x_{3,1} = 0$$

$$x_{1,3} = (c_3 + c_4^2) \cdot x_{3,1} = 0$$

$$x_{0,4} = (c_3 + c_4^2) \cdot x_{2,2} = 0$$

$$x_{1,2} = c_3 \cdot x_{3,0} + c_4 \cdot x_{2,1} + c_5 \cdot x_{3,1} = c_3 \cdot x_{3,0} + c_4 \cdot x_{2,1}$$

$$\begin{aligned} x_{0,3} &= c_3 \cdot x_{2,1} + c_4 \cdot x_{1,2} + c_5 \cdot x_{3,1} = c_3 \cdot x_{2,1} + c_4 \cdot x_{1,2} \\ &= c_3 \cdot x_{2,1} + c_4(c_3 \cdot x_{3,0} + c_4 \cdot x_{2,1}) = c_3c_4 \cdot x_{3,0} + (c_3 + c_4^2) \cdot x_{2,1}. \end{aligned}$$

Note that these are the same equations we obtained when we were investigating the centralizer of $V_2$. Thus, $C(V_3) = C(V_2)$ and $\dim(C(V_3)) - \dim(V_3) = 2$.

Case 3.2: Fix scalars $c_5, c_6 \in \mathbb{Z}_5$ and define

$$m_4 = \begin{pmatrix} 0 & 0 & c_5 & c_3c_4 + c_6(c_3 + c_4^2) & 0 \\ 0 & 0 & c_3 + c_4c_6 & 0 & 0 \\ 0 & c_6 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_3 = \langle V_0, m_2, m_3, m_4 \rangle$. Note that this matrix is formed by multiplying an arbitrary scalar $c_6$ by $v_2$ and adding it to $v_1$, except that we allow the (0,2)-entry to be an arbitrary scalar $c_5$ . It is clear that Cases 3.1 and 3.2 cover all possibilities for the different forms that $m_4$ can take.

Let us now compute the centralizer of $V_3$. By Lemma 14 in Chapter II, each element of $C(V_3)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_3)$ if and only if $\partial_1(x) \in V_3$ and $\partial_2(x) \in V_3$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & 0 & 0 \\ x_{2,1} & x_{2,2} & 0 & 0 & 0 \\ x_{3,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_3$ if and only if

$$x_{1,2} = c_3 \cdot x_{3,0} + c_4 \cdot x_{2,1} + c_5 \cdot x_{4,0}$$

$$x_{3,1} = c_6 \cdot x_{4,0}$$

$$x_{2,2} = (c_3 + c_4 c_6) \cdot x_{4,0}$$

$$x_{1,3} = (c_3 c_4 + c_6(c_3 + c_4^2)) \cdot x_{4,0},$$

and that $\partial_2(x) \in V_3$ if and only if

$$x_{0,3} = c_3 \cdot x_{2,1} + c_4 \cdot x_{1,2} + c_5 \cdot x_{3,1}$$

$$x_{2,2} = c_6 \cdot x_{3,1}$$

$$x_{1,3} = (c_3 + c_4 c_6) \cdot x_{3,1}$$

$$x_{0,4} = (c_3 c_4 + c_6(c_3 + c_4^2)) \cdot x_{3,1}.$$

Hence, $x \in C(V_3)$ if and only if

$$
\begin{aligned}
x_{1,2} &= c_3 \cdot x_{3,0} + c_4 \cdot x_{2,1} + c_5 \cdot x_{4,0} \\
x_{3,1} &= c_6 \cdot x_{4,0} \\
x_{2,2} &= (c_3 + c_4c_6) \cdot x_{4,0} \\
x_{2,2} &= c_6 \cdot x_{3,1} = c_6^2 \cdot x_{4,0} \\
x_{1,3} &= (c_3c_4 + c_6(c_3 + c_4^2)) \cdot x_{4,0} \\
x_{1,3} &= (c_3 + c_4c_6) \cdot x_{3,1} = c_6(c_3 + c_4c_6) \cdot x_{4,0} \\
x_{0,4} &= (c_3c_4 + c_6(c_3 + c_4^2)) \cdot x_{3,1} = c_6(c_3c_4 + c_6(c_3 + c_4^2)) \cdot x_{4,0} \\
x_{0,3} &= c_3 \cdot x_{2,1} + c_4 \cdot x_{1,2} + c_5 \cdot x_{3,1} \\
&= c_3 \cdot x_{2,1} + c_4(c_3 \cdot x_{3,0} + c_4 \cdot x_{2,1} + c_5 \cdot x_{4,0}) + c_5c_6 \cdot x_{4,0} \\
&= c_3c_4 \cdot x_{3,0} + (c_3 + c_4^2) \cdot x_{2,1} + c_5(c_4 + c_6)x_{4,0}.
\end{aligned}
$$

Note that each of $x_{2,2}$ and $x_{1,3}$ has two representations. This yields the following equations:

$$
\begin{aligned}
0 &= (c_6^2 - (c_3 + c_4c_6)) \cdot x_{4,0} \\
0 &= (c_6(c_3 + c_4c_6) - (c_3c_4 + c_6(c_3 + c_4^2))) \cdot x_{4,0}.
\end{aligned}
$$

Observe that if either $c_6^2 \neq (c_3 + c_4c_6)$ or $c_6(c_3 + c_4c_6) \neq (c_3c_4 + c_6(c_3 + c_4^2))$, then

$x_{4,0} = 0$. Note that if $x_{4,0} = 0$, then we have:

$$x_{1,2} = c_3 \cdot x_{3,0} + c_4 \cdot x_{2,1} + c_5 \cdot x_{4,0} = c_3 \cdot x_{3,0} + c_4 \cdot x_{2,1}$$

$$x_{3,1} = c_6 \cdot x_{4,0} = 0$$

$$x_{2,2} = (c_3 + c_4 c_6) \cdot x_{4,0} = 0$$

$$x_{1,3} = (c_3 c_4 + c_6(c_3 + c_4^2)) \cdot x_{4,0} = 0$$

$$x_{0,4} = c_6(c_3 c_4 + c_6(c_3 + c_4^2)) \cdot x_{4,0} = 0$$

$$x_{0,3} = c_3 c_4 \cdot x_{3,0} + (c_3 + c_4^2) \cdot x_{2,1} + c_5(c_4 + c_6) x_{4,0} = c_3 c_4 \cdot x_{3,0} + (c_3 + c_4^2) \cdot x_{2,1}.$$

Note that these are these same equations we obtained when we examined the centralizer of $V_2$. This implies that $C(V_3) = C(V_2)$ and $\dim(C(V_3)) - \dim(V_3) = 2$. Now suppose that $c_6^2 = (c_3 + c_4 c_6)$ and $c_6(c_3 + c_4 c_6) = (c_3 c_4 + c_6(c_3 + c_4^2))$. Note that this would mean that the constants on the 4th anti-diagonal of $m_4$ would be geometric. Further, we have

$$c_6(c_3 + c_4 c_6) = c_6^3$$

$$c_6(c_3 c_4 + c_6(c_3 + c_4^2)) = c_6^2(c_3 + c_4 c_6) = c_6^4.$$

These equations then imply

$$x_{1,2} = c_3 \cdot x_{3,0} + c_4 \cdot x_{2,1} + c_5 \cdot x_{4,0}$$

$$x_{3,1} = c_6 \cdot x_{4,0}$$

$$x_{2,2} = (c_3 + c_4 c_6) \cdot x_{4,0} = c_6^2 \cdot x_{4,0}$$

$$x_{1,3} = (c_3 c_4 + c_6(c_3 + c_4^2)) \cdot x_{4,0} = c_6^3 \cdot x_{4,0}$$

$$x_{0,4} = c_6(c_3 c_4 + c_6(c_3 + c_4^2)) \cdot x_{4,0} = c_6^4 \cdot x_{4,0}$$

$$x_{0,3} = c_3 c_4 \cdot x_{3,0} + (c_3 + c_4^2) \cdot x_{2,1} + c_5(c_4 + c_6) x_{4,0}.$$

We define

$$v_3 = \begin{pmatrix} 0 & 0 & 0 & c_5(c_4+c_6) & c_6^4 \\ 0 & 0 & c_5 & c_6^3 & 0 \\ 0 & 1 & c_6^2 & 0 & 0 \\ 0 & c_6 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and note that $C(V_3) = \langle V(\beta_2), v_1, v_2, v_3 \rangle$. Notice once again that the entries on the anti-diagonal form a geometric sequence. This is as far as this case has been investigated. We shall now go back to Case 1.1.

Case 1.1: Recall that $V_1 = \langle V_0, m_1 \rangle$ and $C(V_1) = \langle V(\beta_2), v_1 \rangle$ where

$$m_1 = \begin{pmatrix} 0 & 0 & c_1^2 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad v_1 = \begin{pmatrix} 0 & 0 & 0 & c_1^3 & 0 \\ 0 & 0 & c_1^2 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Recall also that $\dim(C(V_1)) - \dim(V_1) = 3$. Hence, we must define a new matrix $m_2 \in C(V_1) - V_1$ and consider $V_2 = \langle V_1, m_2 \rangle$. In Case 3, we examined what happens when the (2,0)-entry, the (1,1)-entry, and the (0,2)-entry are the only non-zero entries $m_1$ and $m_2$. Hence, to avoid repetition of earlier cases, the coefficient on $v_1$ in the expansion of $m_2$ must be nonnegative. To this end, fix $c_3, c_4 \in \mathbb{Z}_5$, we define

$$m_2 = \begin{pmatrix} 0 & 0 & c_4 & c_1^3 & 0 \\ 0 & c_3 & c_1^2 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_2 = \langle V_0, m_1, m_2 \rangle$.

Let us now compute the centralizer of $V_2$. By Lemma 14 in Chapter II, each

element of $C(V_2)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_2)$ if and only if $\partial_1(x) \in V_2$ and $\partial_2(x) \in V_2$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & 0 & 0 \\ x_{2,1} & x_{2,2} & 0 & 0 & 0 \\ x_{3,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_2$ if and only if

$$x_{2,1} = c_1 \cdot x_{3,0} + c_3 \cdot x_{4,0}$$

$$x_{1,2} = c_1^2 \cdot x_{3,0} + c_4 \cdot x_{4,0}$$

$$x_{3,1} = c_1 \cdot x_{4,0}$$

$$x_{2,2} = c_1^2 \cdot x_{4,0}$$

$$x_{1,3} = c_1^3 \cdot x_{4,0},$$

and that $\partial_2(x) \in V_2$ if and only if

$$x_{1,2} = c_1 \cdot x_{2,1} + c_3 \cdot x_{3,1}$$

$$x_{0,3} = c_1^2 \cdot x_{2,1} + c_4 \cdot x_{3,1}$$

$$x_{2,2} = c_1 \cdot x_{3,1}$$

$$x_{1,3} = c_1^2 \cdot x_{3,1}$$

$$x_{0,4} = c_1^3 \cdot x_{3,1}.$$

Hence, $x \in C(V_2)$ if and only if

$$\begin{aligned}
x_{2,1} &= c_1 \cdot x_{3,0} + c_3 \cdot x_{4,0} \\
x_{3,1} &= c_1 \cdot x_{4,0} \\
x_{1,2} &= c_1^2 \cdot x_{3,0} + c_4 \cdot x_{4,0} \\
x_{1,2} &= c_1 \cdot x_{2,1} + c_3 \cdot x_{3,1} \\
&= c_1(c_1 \cdot x_{3,0} + c_3 \cdot x_{4,0}) + c_1 c_3 \cdot x_{4,0} \\
&= c_1^2 \cdot x_{3,0} + 2c_1 c_3 \cdot x_{4,0} \\
x_{0,3} &= c_1^2 \cdot x_{2,1} + c_4 \cdot x_{3,1} \\
&= c_1^2(c_1 \cdot x_{3,0} + c_3 \cdot x_{4,0}) + c_1 c_4 \cdot x_{4,0} \\
&= c_1^3 \cdot x_{3,0} + (c_1^2 c_3 + c_1 c_4)x_{4,0} \\
x_{2,2} &= c_1 \cdot x_{3,1} = c_1^2 \cdot x_{4,0} \\
x_{1,3} &= c_1^2 \cdot x_{3,1} = c_1^3 \cdot x_{4,0} \\
x_{0,4} &= c_1^3 \cdot x_{3,1} = c_1^4 \cdot x_{4,0}.
\end{aligned}$$

Note that the third and fourth of the above equations give us the equation $(c_4 - 2c_1c_3)x_{4,0} = 0$. In the case where $c_4 \neq 2c_1c_3$ we have that $x_{4,0} = 0$. This would give us the following equations:

$$x_{2,1} = c_1 \cdot x_{3,0} + c_3 \cdot x_{4,0} = c_1 \cdot x_{3,0}$$

$$x_{1,2} = c_1^2 \cdot x_{3,0} + c_4 \cdot x_{4,0} = c_1^2 \cdot x_{3,0}$$

$$x_{0,3} = c_1^3 \cdot x_{3,0} + (c_1^2c_3 + c_1c_4)x_{4,0} = c_1^3 \cdot x_{3,0}$$

$$x_{3,1} = c_1 \cdot x_{4,0} = 0$$

$$x_{2,2} = c_1^2 \cdot x_{4,0} = 0$$

$$x_{1,3} = c_1^3 \cdot x_{4,0} = 0$$

$$x_{0,4} = c_1^4 \cdot x_{4,0} = 0.$$

Note that these are the same equations which we obtained when we examined the centralizer of $V_1$. This would imply that $C(V_2) = C(V_1)$ and $\dim(C(V_2)) - \dim(V_2) = 2$.

In case $c_4 = 2c_1c_3$, we have the following equations:

$$\begin{aligned}
x_{2,1} &= c_1 \cdot x_{3,0} + c_3 \cdot x_{4,0} \\
x_{1,2} &= c_1^2 \cdot x_{3,0} + c_4 \cdot x_{4,0} \\
x_{0,3} &= c_1^3 \cdot x_{3,0} + (c_1^2 c_3 + c_1 c_4) x_{4,0} \\
&= c_1^3 \cdot x_{3,0} + (c_1^2 c_3 + 2c_1^2 c_3) x_{4,0} \\
&= c_1^3 \cdot x_{3,0} + 3c_1^2 c_3 \cdot x_{4,0} \\
x_{3,1} &= c_1 \cdot x_{4,0} \\
x_{2,2} &= c_1^2 \cdot x_{4,0} \\
x_{1,3} &= c_1^3 \cdot x_{4,0} \\
x_{0,4} &= c_1^4 \cdot x_{4,0}.
\end{aligned}$$

Thus, we define

$$v_2 = \begin{pmatrix} 0 & 0 & 0 & 3c_1^2 c_3 & c_1^4 \\ 0 & 0 & c_4 & c_1^3 & 0 \\ 0 & c_3 & c_1^2 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and note that $C(V_2) = \langle V(\beta_2), v_1, v_2 \rangle$. Notice once again that the entries on the anti-diagonal form a geometric sequence. This is a theme we will see again when examining the next pattern. However, in the final pattern, there will be an interesting twist. This is as far as the present pattern has been investigated.

## 6.3 The Third Wave Pattern

This pattern will not be fully explored. After one iteration of our algorithm, we will define Case 1, Case 2, and Case 3 since there will be three different ways to define the matrix $m_1$. Case 1 will be further subdivided into Case 1.1 and Case 1.2 on the basis of a possible relationship between the entries of the matrix $m_1$. We shall iterate our algorithm once for Case 1.2 and then cease investigating it. After one iteration of our algorithm for Cases 1.1, 2, and 3, we will be able to treat them simultaneously. Hence, we will combine them into a single case, which we will call Case 4. Case 4 will then be subdivided into Case 4.1, Case 4.2, and Case 4.3 since there will be three different ways to define the matrices $m_2$ and $m_3$. Case 4.1 will further be subdivided into Case 4.1.1, Case 4.1.2, Case 4.1.3, and Case 4.1.4 due to different possible relationships among the coefficients of $m_2$ and $m_3$. We shall iterate our algorithm once on Cases 4.1.1 through 4.1.4, Case 4.2, and Case 4.3 and then cease investigating them. Case 5 will be derived from a subset of the subcases of Case 4. Case 5 shall be subdivided into Case 5.1, Case 5.2, and Case 5.3 due to the different ways one will be able to define the matrix $m_7$. To illustrate some of the recurring themes of these results, after we iterate our algorithm once on Cases 5.1 through 5.3, we will examine "similar looking" subspaces under the assumption that $p = 7$. Investigation of this pattern will cease after this.

We define the pattern

$$\alpha_3 = \begin{pmatrix} \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and write $V_0 = V(\alpha_3)$. Clearly, $C(V) = V(\beta_3)$ where

$$\beta_3 = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Case 1: Fix scalars $c_1, c_2, c_3 \in \mathbb{Z}_5$ such that $(c_1, c_2, c_3) \neq (0, 0, 0)$ and define

$$m_1 = \begin{pmatrix} 0 & 0 & 0 & c_3 & 0 \\ 0 & 0 & c_2 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_1 = \langle V_0, m_1 \rangle$.

Let us now compute the centralizer of $V_1$. By Lemma 14 in Chapter II, each element of $C(V_1)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_1)$ if and only if $\partial_1(x) \in V_1$ and $\partial_2(x) \in V_1$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & 0 & 0 \\ x_{2,1} & x_{2,2} & 0 & 0 & 0 \\ x_{3,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_1$ if and only if

$$x_{3,1} = c_1 \cdot x_{4,0}$$

$$x_{2,2} = c_2 \cdot x_{4,0}$$

$$x_{1,3} = c_3 \cdot x_{4,0},$$

and that $\partial_2(x) \in V_1$ if and only if

$$x_{2,2} = c_1 \cdot x_{3,1}$$

$$x_{1,3} = c_2 \cdot x_{3,1}$$

$$x_{0,4} = c_3 \cdot x_{3,1}.$$

Hence, $x \in C(V_1)$ if and only if

$$x_{3,1} = c_1 \cdot x_{4,0}$$

$$x_{2,2} = c_2 \cdot x_{4,0} = c_1 \cdot x_{3,1} = c_1^2 \cdot x_{4,0}$$

$$x_{1,3} = c_3 \cdot x_{4,0} = c_2 \cdot x_{3,1} = c_1 c_2 \cdot x_{4,0}$$

$$x_{0,4} = c_3 \cdot x_{3,1}.$$

Note that the above equations imply the following equations:

$$(c_2 - c_1^2)x_{4,0} = 0$$

$$(c_3 - c_1 c_2)x_{4,0} = 0.$$

Thus we see that if either $c_2 \neq c_1^2$ or $c_3 \neq c_1 c_2$ then it must be the case that $x_{4,0} = 0$. This would then imply that $x_{4,0} = x_{3,1} = x_{2,2} = x_{1,3} = x_{0,4} = 0$ and $C(V_1) = C(V_0)$. This would further imply that $\dim(C(V_1)) - \dim(V_1) = 3$. This will be regarded as Case 1.1 and shall be investigated later in the thesis.

However, if $c_2 = c_1^2$ and $c_3 = c_1 c_2$, then we have that $c_3 = c_1 c_2 = c_1 c_1^2 = c_1^3$ and

$$x_{3,1} = c_1 \cdot x_{4,0}$$

$$x_{2,2} = c_1^2 \cdot x_{4,0}$$

$$x_{1,3} = c_1^3 \cdot x_{4,0}$$

$$x_{0,4} = c_3 \cdot x_{3,1} = c_1^3 c_1 \cdot x_{4,0} = c_1^4 \cdot x_{4,0}.$$

Thus, we define

$$v_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_1^4 \\ 0 & 0 & 0 & c_1^3 & 0 \\ 0 & 0 & c_1^2 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and note that $C(V_1) = \langle V(\beta_3), v_1 \rangle$. This shall be regarded as Case 1.2.

Case 1.2: We must add another matrix to our current span. Later on we shall consider the case where our new matrix contains non-zero entries only on the fourth anti-diagonal. Hence, we shall make use of $v_1$ in forming our new matrix. To this end, fix $c_4, c_5, c_6 \in \mathbb{Z}_5$ and define

$$m_2 = \begin{pmatrix} 0 & 0 & 0 & c_6 & c_1^4 \\ 0 & 0 & c_5 & c_1^3 & 0 \\ 0 & c_4 & c_1^2 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $V_2 = \langle V_1, m_2 \rangle$.

We shall now compute the centralizer of $V_2$. By Lemma 14 in Chapter II,

each element of $C(V_2)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_2)$ if and only if $\partial_1(x) \in V_2$ and $\partial_2(x) \in V_2$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_2$ if and only if

$$x_{3,1} = c_1 \cdot x_{4,0}$$

$$x_{2,2} = c_2 \cdot x_{4,0}$$

$$x_{1,3} = c_3 \cdot x_{4,0}$$

$$x_{4,1} = x_{3,2} = x_{2,3} = x_{1,4} = 0,$$

and that $\partial_2(x) \in V_2$ if and only if

$$x_{2,2} = c_1 \cdot x_{3,1} + c_4 \cdot x_{4,1} = c_1 \cdot x_{3,1}$$

$$x_{1,3} = c_2 \cdot x_{3,1} + c_5 \cdot x_{4,1} = c_2 \cdot x_{3,1}$$

$$x_{0,4} = c_3 \cdot x_{3,1} + c_6 \cdot x_{4,1} = c_3 \cdot x_{3,1}$$

$$x_{3,2} = c_1 \cdot x_{4,1} = 0$$

$$x_{2,3} = c_1^2 \cdot x_{4,1} = 0$$

$$x_{3,2} = c_1^3 \cdot x_{4,1} = 0$$

$$0 = c_1^4 \cdot x_{4,1}.$$

Note that these are the same equations we obtained when we computed the centralizer of $V_1$. Thus, $C(V_2) = C(V_1)$ and $\dim(C(V_2)) - \dim(V_2) = 3$. Note that even though $m_2$ has anti-diagonal entries which form a geometric sequence, it is still the case that $C(V_2) = C(V_1)$. This is due to the fact that the coefficient on $m_2$ in the expansion of $\partial_1(x)$ is 0 since the (4,0)-entry of $m_2$ is non-zero. Keep this in mind as this theme will repeat itself. This is as far as Case 1.2 has been taken.

Case 2: Fix $c_1, c_2 \in \mathbb{Z}_5$ such that $(c_1, c_2) \neq (0, 0)$ and define

$$m_1 = \begin{pmatrix} 0 & 0 & 0 & c_2 & 0 \\ 0 & 0 & c_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_1 = \langle V_0, m_1 \rangle$.

We shall now compute the centralizer of $V_1$. By Lemma 14 in Chapter II, each element of $C(V_1)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_1)$ if and only if $\partial_1(x) \in V_1$ and $\partial_2(x) \in V_1$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & 0 & 0 \\ x_{2,1} & x_{2,2} & 0 & 0 & 0 \\ x_{3,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_1$ if and only if

$$x_{4,0} = 0$$

$$x_{2,2} = c_1 \cdot x_{3,1}$$

$$x_{1,3} = c_2 \cdot x_{3,1},$$

and that $\partial_2(x) \in V_1$ if and only if

$$x_{3,1} = 0$$

$$x_{1,3} = c_1 \cdot x_{2,2}$$

$$x_{0,4} = c_2 \cdot x_{2,2}.$$

Hence, $x \in C(V_1)$ if and only if

$$x_{4,0} = 0$$

$$x_{3,1} = 0$$

$$x_{2,2} = c_1 \cdot x_{3,1} = 0$$

$$x_{1,3} = c_2 \cdot x_{3,1} = 0$$

$$x_{0,4} = c_2 \cdot x_{2,2} = 0.$$

Thus, $C(V_1) = C(V_0)$ which implies $\dim(C(V_1)) - \dim(V_1) = 3$. We shall return to this case later in the thesis.

Case 3: Fix a non-zero scalar $c_1 \in \mathbb{Z}_5$ and define

$$m_1 = \begin{pmatrix} 0 & 0 & 0 & c_1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_1 = \langle V_0, m_1 \rangle$.

Let us now compute the centralizer of $V_1$. By Lemma 14 in Chapter II, each

element of $C(V_1)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_1)$ if and only if $\partial_1(x) \in V_1$ and $\partial_2(x) \in V_1$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & 0 & 0 \\ x_{2,1} & x_{2,2} & 0 & 0 & 0 \\ x_{3,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_1$ if and only if

$$x_{4,0} = 0$$

$$x_{3,1} = 0$$

$$x_{1,3} = c_1 \cdot x_{2,2},$$

and that $\partial_2(x) \in V_1$ if and only if

$$x_{3,1} = 0$$

$$x_{2,2} = 0$$

$$x_{0,4} = c_1 \cdot x_{1,3}.$$

Hence, $x \in C(V_1)$ if and only if

$$x_{4,0} = 0$$

$$x_{3,1} = 0$$

$$x_{2,2} = 0$$

$$x_{1,3} = c_1 \cdot x_{2,2} = 0$$

$$x_{0,4} = c_1 \cdot x_{1,3} = 0.$$

Thus, $C(V_1) = C(V_0)$ and $\dim(C(V_1)) - \dim(V_1) = 3$.

Notice that Cases 2, 3, and 1.1 all end the same way. Namely, $C(V_1) = C(V_0)$. Keep these results in mind as they represent a theme we will see throughout this thesis. In any event we shall regard these cases as a single case, Case 4.

Case 4: We now seek the 2-dimensional good subspaces of $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. There are three different ways we can do this depending on what our set of leading positions is. That is, our set of leading positions can either be $\{1, 2\}$, $\{1, 3\}$, or $\{2, 3\}$. As such, we have three subcases to consider.

Case 4.1: Suppose that our set of leading positions is $\{1, 2\}$, fix $c_4, c_5, c_6, c_7 \in \mathbb{Z}_5$ such that $(c_4, c_5) \neq (0, 0)$ and $(c_6, c_7) \neq (0, 0)$, and define

$$m_2 = \begin{pmatrix} 0 & 0 & 0 & c_5 & 0 \\ 0 & 0 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad m_3 = \begin{pmatrix} 0 & 0 & 0 & c_7 & 0 \\ 0 & 0 & c_6 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_2 = \langle V_0, m_2, m_3 \rangle$.

Let us now compute the centralizer of $V_2$. By Lemma 14 in Chapter II, each element of $C(V_2)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_2)$ if and only if $\partial_1(x) \in V_2$ and $\partial_2(x) \in V_2$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & 0 & 0 \\ x_{2,1} & x_{2,2} & 0 & 0 & 0 \\ x_{3,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_2$ if and only if

$$x_{2,2} = c_4 \cdot x_{4,0} + c_6 \cdot x_{3,1}$$

$$x_{1,3} = c_5 \cdot x_{4,0} + c_7 \cdot x_{3,1},$$

and that $\partial_2(x) \in V_2$ if and only if

$$x_{1,3} = c_4 \cdot x_{3,1} + c_6 \cdot x_{2,2}$$

$$x_{0,4} = c_5 \cdot x_{3,1} + c_7 \cdot x_{2,2}.$$

Hence, $x \in C(V_2)$ if and only if

$$\begin{aligned} x_{2,2} &= c_4 \cdot x_{4,0} + c_6 \cdot x_{3,1} \\ x_{1,3} &= c_5 \cdot x_{4,0} + c_7 \cdot x_{3,1} = c_4 \cdot x_{3,1} + c_6 \cdot x_{2,2} \\ &= c_4 \cdot x_{3,1} + c_6(c_4 \cdot x_{4,0} + c_6 \cdot x_{3,1}) = (c_4 + c_6^2)x_{3,1} + c_4c_6x_{4,0} \\ x_{0,4} &= c_5 \cdot x_{3,1} + c_7 \cdot x_{2,2} = c_5 \cdot x_{3,1} + c_7(c_4 \cdot x_{4,0} + c_6 \cdot x_{3,1}) \\ &= (c_5 + c_6c_7)x_{3,1} + c_4c_7 \cdot x_{4,0}. \end{aligned}$$

Note that the above equations imply the following equation:

$$(c_5 - c_4c_6) \cdot x_{4,0} + (c_7 - (c_4 + c_6^2)) \cdot x_{3,1} = 0.$$

This leads to four more subcases of Case 4.1.

Case 4.1.1: Suppose that $c_5 = c_4c_6$ and $c_7 = (c_4 + c_6^2)$. Then

$$\begin{aligned} x_{2,2} &= c_4 \cdot x_{4,0} + c_6 \cdot x_{3,1} \\ x_{1,3} &= c_4c_6x_{4,0} + (c_4 + c_6^2)x_{3,1} \\ x_{0,4} &= c_4c_7 \cdot x_{4,0} + (c_5 + c_6c_7)x_{3,1} \\ &= c_4(c_4 + c_6^2)x_{4,0} + (c_4c_6 + c_6(c_4 + c_6^2))x_{3,1} \\ &= c_4(c_4 + c_6^2)x_{4,0} + (2c_4c_6 + c_6^3)x_{3,1}. \end{aligned}$$

Hence, we define,

$$v_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_4c_7 \\ 0 & 0 & 0 & c_4c_6 & 0 \\ 0 & 0 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_4c_6 + c_5 + c_6^3 \\ 0 & 0 & 0 & c_4 + c_6^2 & 0 \\ 0 & 0 & c_6 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and note that $C(V_2) = \langle V(\beta_3), v_1, v_2\rangle$ and $\dim(C(V_2)) - \dim(V_2) = 4$.

Case 4.1.2: Now $c_5 = c_4c_6$ but $c_7 \neq (c_4 + c_6^2)$. This implies that

$$(c_5 - c_4c_6) \cdot x_{4,0} + (c_7 - (c_4 + c_6^2)) \cdot x_{3,1} = (c_7 - (c_4 + c_6^2) \cdot x_{3,1} = 0.$$

This further implies that $x_{3,1} = 0$. Hence, we now have the following equations:

$$x_{2,2} = c_4 \cdot x_{4,0} + c_6 \cdot x_{3,1} = c_4 \cdot x_{4,0}$$

$$x_{1,3} = (c_4 + c_6^2)x_{3,1} + c_4c_6x_{4,0} = c_4c_6 \cdot x_{4,0}$$

$$x_{0,4} = (c_5 + c_6c_7)x_{3,1} + c_4c_7 \cdot x_{4,0} = c_4c_7 \cdot x_{4,0}.$$

Hence, $C(V_2) = \langle V(\beta_3), v_1\rangle$ where $v_1$ is the matrix from Case 4.1.1. Also note that $\dim(C(V_2)) - \dim(V_2) = 3$.

Case 4.1.3: Now assume $c_5 \neq c_4c_6$ but $c_7 = (c_4 + c_6^2)$. This implies that

$$(c_5 - c_4c_6) \cdot x_{4,0} + (c_7 - (c_4 + c_6^2)) \cdot x_{3,1} = (c_5 - c_4c_6) \cdot x_{4,0} = 0.$$

This further implies that $x_{4,0} = 0$. Hence, we now have the following:

$$x_{2,2} = c_4 \cdot x_{4,0} + c_6 \cdot x_{3,1} = c_6 \cdot x_{3,1}$$

$$x_{1,3} = (c_4 + c_6^2)x_{3,1} + c_4c_6x_{4,0} = (c_4 + c_6^2)x_{3,1}$$

$$x_{0,4} = (c_5 + c_6c_7)x_{3,1} + c_4c_7 \cdot x_{4,0} = (c_5 + c_6c_7)x_{3,1}$$

$$= (c_5 + c_6(c_4 + c_6^2))x_{3,1} = (c_4c_6 + c_5 + c_6^3)x_{3,1}.$$

Hence, $C(V_2) = \langle V(\beta_3), v_2 \rangle$ where $v_2$ is the matrix from Case 4.1.1. Also note that $\dim(C(V_2)) - \dim(V_2) = 3$.

Case 4.1.4: Now assume that $c_5 \neq c_4c_6$ and $c_7 \neq (c_4 + c_6^2)$. In this case, since we have

$$(c_5 - c_4c_6) \cdot x_{4,0} + (c_7 - (c_4 + c_6^2)) \cdot x_{3,1} = 0,$$

we can explicitly solve for $x_{3,1}$ in terms of $x_{4,0}$. For simplicity of presentation, we define $d = \frac{c_5 - c_4c_6}{(c_4+c_6^2)-c_7}$ and obtain

$$x_{3,1} = d \cdot x_{4,0}$$

$$x_{2,2} = (c_4 + c_6d) \cdot x_{4,0}$$

$$x_{1,3} = (c_4 + c_6^2)x_{3,1} + c_4c_6x_{4,0} = (c_4c_6 + (c_4 + c_6^2)d) \cdot x_{4,0}$$

$$x_{0,4} = (c_5 + c_6c_7)x_{3,1} + c_4c_7 \cdot x_{4,0} = (c_4c_7 + (c_5 + c_6c_7)d) \cdot x_{4,0}.$$

We now define

$$v_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_4c_7 + (c_5 + c_6c_7)d \\ 0 & 0 & 0 & c_4c_6 + (c_4 + c_6^2)d & 0 \\ 0 & 0 & c_4 + c_6d & 0 & 0 \\ 0 & d & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and note that $C(V_2) = \langle V(\beta_3), v_1 \rangle$ and $\dim(C(V_2)) - \dim(V_2) = 3$. This is as far as this case has been explored.

Case 4.2: Suppose that our set of leading positions is $\{1, 3\}$ and fix $c_4, c_5, c_6 \in \mathbb{Z}_5$ such that $(c_4, c_5) \neq (0, 0)$ and $c_6 \neq 0$. We define

$$m_2 = \begin{pmatrix} 0 & 0 & 0 & c_5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & c_4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad m_3 = \begin{pmatrix} 0 & 0 & 0 & c_6 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_2 = \langle V_0, m_1, m_2 \rangle$.

Let us now compute the centralizer of $V_2$. By Lemma 14 in Chapter II, each

element of $C(V_2)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_2)$ if and only if $\partial_1(x) \in V_2$ and $\partial_2(x) \in V_2$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & 0 & 0 \\ x_{2,1} & x_{2,2} & 0 & 0 & 0 \\ x_{3,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_2$ if and only if

$$x_{3,1} = c_4 \cdot x_{4,0}$$

$$x_{1,3} = c_5 \cdot x_{4,0} + c_6 \cdot x_{2,2},$$

and that $\partial_2(x) \in V_2$ if and only if

$$x_{2,2} = c_4 \cdot x_{3,1}$$

$$x_{0,4} = c_5 \cdot x_{3,1} + c_6 \cdot x_{1,3}.$$

Hence, $x \in C(V_2)$ if and only if

$$x_{3,1} = c_4 \cdot x_{4,0}$$

$$x_{2,2} = c_4 \cdot x_{3,1} = c_4^2 \cdot x_{4,0}$$

$$x_{1,3} = c_5 \cdot x_{4,0} + c_6 \cdot x_{2,2} = (c_5 + c_4^2 c_6) x_{4,0}$$

$$x_{0,4} = c_5 \cdot x_{3,1} + c_6 \cdot x_{1,3} = (c_4 c_5 + c_6(c_5 + c_4^2 c_6)) x_{4,0}.$$

Thus, we define

$$v_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_4c_5 + c_6(c_5 + c_4^2 c_6) \\ 0 & 0 & 0 & c_5 + c_4^2 c_6 & 0 \\ 0 & 0 & c_4^2 & 0 & 0 \\ 0 & c_4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and note that $C(V_2) = \langle V(\beta_3), v_1 \rangle$ and $\dim(C(V_2)) - \dim(V_2) = 3$. This is as far as this case has been examined.

Case 4.3: Suppose that our set of leading positions is $\{2, 3\}$ and fix $c_4, c_5 \in \mathbb{Z}_5$ such that $c_4 \neq 0$ and $c_5 \neq 0$. We define

$$m_2 = \begin{pmatrix} 0 & 0 & 0 & c_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad m_3 = \begin{pmatrix} 0 & 0 & 0 & c_5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and write $V_2 = \langle V, m_1, m_2 \rangle$.

Let us now compute the centralizer of $V_2$. By Lemma 14 in Chapter II, each element of $C(V_2)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_2)$ if and only if $\partial_1(x) \in V_2$ and $\partial_2(x) \in V_2$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & 0 & 0 \\ x_{2,1} & x_{2,2} & 0 & 0 & 0 \\ x_{3,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_2$ if and only if

$$x_{4,0} = 0$$

$$x_{1,3} = c_4 \cdot x_{3,1} + c_5 \cdot x_{2,2},$$

and that $\partial_2(x) \in V_2$ if and only if

$$x_{3,1} = 0$$

$$x_{0,4} = c_4 \cdot x_{2,2} + c_5 \cdot x_{1,3}.$$

Hence, $x \in C(V_2)$ if and only if

$$x_{4,0} = 0$$

$$x_{3,1} = 0$$

$$x_{1,3} = c_4 \cdot x_{3,1} + c_5 \cdot x_{2,2} = c_5 \cdot x_{2,2}$$

$$x_{0,4} = c_4 \cdot x_{2,2} + c_5 \cdot x_{1,3} = c_4 \cdot x_{2,2} + c_5^2 \cdot x_{2,2} = (c_4 + c_5^2)x_{2,2}.$$

Thus, we define

$$v_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_4 + c_5^2 \\ 0 & 0 & 0 & c_5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and note that $C(V_2) = \langle V(\beta_3), v_1 \rangle$ and $\dim(C(V_2)) - \dim(V_2) = 3$. This is as far as this case has been examined.

Case 5: We shall now investigate what occurs when we consider the three-dimensional good subspaces of $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$. Using Theorem 6 in Chapter III, we know that there is only one form such spaces may take. To this end, fix non-zero scalars $c_8, c_9, c_{10} \in \mathbb{Z}_5$ and define

$$m_4 = \begin{pmatrix} 0 & 0 & 0 & c_8 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad m_5 = \begin{pmatrix} 0 & 0 & 0 & c_9 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$m_6 = \begin{pmatrix} 0 & 0 & 0 & c_{10} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $V_3 = \langle V_0, m_4, m_5, m_6 \rangle$.

Let us now compute the centralizer of $V_3$. By Lemma 14 in Chapter II, each element of $C(V_3)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_3)$ if and only if $\partial_1(x) \in V_3$ and $\partial_2(x) \in V_3$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & 0 & 0 \\ x_{3,0} & x_{3,1} & 0 & 0 & 0 \\ x_{4,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & 0 & 0 \\ x_{2,1} & x_{2,2} & 0 & 0 & 0 \\ x_{3,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_3$ if and only if

$$x_{1,3} = c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2},$$

and that $\partial_2(x) \in V_3$ if and only if

$$x_{0,4} = c_8 \cdot x_{3,1} + c_9 \cdot x_{2,2} + c_{10} \cdot x_{1,3}.$$

Hence, $x \in C(V_3)$ if and only if

$$\begin{aligned} x_{1,3} &= c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2} \\ x_{0,4} &= c_8 \cdot x_{3,1} + c_9 \cdot x_{2,2} + c_{10} \cdot x_{1,3} \\ &= c_8 \cdot x_{3,1} + c_9 \cdot x_{2,2} + c_{10}(c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2}) \\ &= c_8 c_{10} \cdot x_{4,0} + (c_8 + c_9 c_{10}) x_{3,1} + (c_9 + c_{10}^2) x_{2,2}. \end{aligned}$$

Hence, we define

$$v_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_8c_{10} \\ 0 & 0 & 0 & c_8 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_8 + c_9c_{10} \\ 0 & 0 & 0 & c_9 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{and} \quad v_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_9 + c_{10}^2 \\ 0 & 0 & 0 & c_{10} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and note that $C(V_3) = \langle V(\beta_3), v_2, v_3, v_4 \rangle$ and $\dim(C(V_3)) - \dim(V_3) = 4$.

Here is where we will do something different from what we have done before. We will still investigate what happens when we put a new matrix in our basis. However, afterward, we will do the same thing except under the assumption that $p = 7$. There are two reasons for doing this. First, it will demonstrate that the centralizer of a given subspace will be different depending on which prime is chosen if one has a basis matrix with non-zero entries in the $(p-1)$th row or the $(p-1)$th column. Second, it will once again show that if $C(V_i) \subset C(V_{i+1})$, then one has a basis matrix whose leading anti-diagonal entries form a geometric sequence.

Case 5.1 ($p = 5$): Fix $c_{11} \in \mathbb{Z}_5$ and define

$$m_7 = \begin{pmatrix} 0 & 0 & 0 & c_{11} & c_9 + c_{10}^2 \\ 0 & 0 & 0 & c_{10} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $V_4 = \langle V_0, m_4, m_5, m_6, m_7 \rangle$.

Let us now compute the centralizer of $V_4$. By Lemma 14 in Chapter II, each element of $C(V_4)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_4)$ if and only if $\partial_1(x) \in V_4$ and $\partial_2(x) \in V_4$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_4$ if and only if

$$x_{1,3} = c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2} + c_{11} \cdot x_{3,2}$$

$$x_{4,1} = 0$$

$$x_{2,3} = c_{10} \cdot x_{3,2}$$

$$x_{1,4} = (c_9 + c_{10}^2)x_{3,2},$$

and that $\partial_2(x) \in V_4$ if and only if

$$x_{0,4} = c_8 \cdot x_{3,1} + c_9 \cdot x_{2,2} + c_{10} \cdot x_{1,3} + c_{11} \cdot x_{2,3}$$

$$x_{3,2} = 0$$

$$x_{1,4} = c_{10} \cdot x_{2,3}$$

$$0 = (c_9 + c_{10}^2)x_{2,3}.$$

Hence, $x \in C(V_4)$ if and only if

$$x_{4,1} = 0$$

$$x_{3,2} = 0$$

$$x_{2,3} = c_{10} \cdot x_{3,2} = 0$$

$$x_{1,4} = c_{10} \cdot x_{2,3} = 0$$

$$\begin{aligned} x_{1,3} &= c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2} + c_{11} \cdot x_{3,2} \\ &= c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2} \end{aligned}$$

$$\begin{aligned} x_{0,4} &= c_8 \cdot x_{3,1} + c_9 \cdot x_{2,2} + c_{10} \cdot x_{1,3} + c_{11} \cdot x_{2,3} \\ &= c_8 \cdot x_{3,1} + c_9 \cdot x_{2,2} + c_{10} \cdot x_{1,3}. \end{aligned}$$

Note that these are the same equations we obtained when we computed $C(V_3)$. Hence, $C(V_4) = C(V_3)$ and $\dim(C(V_4)) - \dim(V_4) = 3$.

Case 5.2 ($p = 5$): Fix $c_{11}, c_{12} \in \mathbb{Z}_5$ and define

$$m_7 = \begin{pmatrix} 0 & 0 & 0 & c_{12} & c_8 + c_9c_{10} + c_{11}(c_9 + c_{10}^2) \\ 0 & 0 & 0 & c_9 + c_{10}c_{11} & 0 \\ 0 & 0 & c_{11} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $V_4 = \langle V_0, m_4, m_5, m_6, m_7 \rangle$.

Let us now compute the centralizer of $V_4$. By Lemma 14 in Chapter II, each element of $C(V_4)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_4)$ if and only if $\partial_1(x) \in V_4$ and $\partial_2(x) \in V_4$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_4$ if and only if

$$x_{1,3} = c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2} + c_{12} \cdot x_{4,1}$$

$$x_{3,2} = c_{11} \cdot x_{4,1}$$

$$x_{2,3} = (c_9 + c_{10}c_{11})x_{4,1}$$

$$x_{1,4} = (c_8 + c_9c_{10} + c_{11}(c_9 + c_{10}^2))x_{4,1},$$

and that $\partial_2(x) \in V_4$ if and only if

$$x_{0,4} = c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2} + c_{12} \cdot x_{3,2}$$

$$x_{4,1} = 0$$

$$x_{2,3} = c_{11} \cdot x_{3,2}$$

$$x_{1,4} = (c_9 + c_{10}c_{11})x_{3,2}$$

$$0 = (c_8 + c_9c_{10} + c_{11}(c_9 + c_{10}^2))x_{3,2}.$$

Hence, $x \in C(V_4)$ if and only if

$$x_{4,1} = 0$$

$$x_{3,2} = c_{11} \cdot x_{4,1} = 0$$

$$x_{2,3} = (c_9 + c_{10}c_{11})x_{4,1} = 0$$

$$x_{1,4} = (c_8 + c_9c_{10} + c_{11}(c_9 + c_{10}^2))x_{4,1} = 0$$

$$x_{1,3} = c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2} + c_{12} \cdot x_{4,1}$$

$$= c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2}$$

$$x_{0,4} = c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2} + c_{12} \cdot x_{3,2}$$
$$= c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2}.$$

Note that these are the same equations we obtained when we computed $C(V_3)$. Hence, $C(V_4) = C(V_3)$ and $\dim(C(V_4)) - \dim(V_4) = 3$.

Case 5.3 ($p = 5$): Fix $c_{11}, c_{12}, c_{13} \in \mathbb{Z}_5$ and define

$$m_7 = \begin{pmatrix} 0 & 0 & 0 & c_{13} & c_8c_{10} + c_{11}(c_8 + c_9c_{10}) + c_{12}(c_9 + c_{10}^2) \\ 0 & 0 & 0 & c_8 + c_9c_{11} + c_{10}c_{12} & 0 \\ 0 & 0 & c_{12} & 0 & 0 \\ 0 & c_{11} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $V_4 = \langle V, m_4, m_5, m_6, m_7 \rangle$.

Let us now compute the centralizer of $V_4$. By Lemma 14 in Chapter II, each element of $C(V_4)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_4)$ if and only if $\partial_1(x) \in V_4$ and $\partial_2(x) \in V_4$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_4$ if and only if

$$\begin{aligned} x_{1,3} &= c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2} \\ x_{4,1} &= 0 \\ x_{3,2} &= 0 \\ x_{2,3} &= 0 \\ x_{1,4} &= 0, \end{aligned}$$

and that $\partial_2(x) \in V_4$ if and only if

$$x_{0,4} = c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2} + c_{13} \cdot x_{3,2} = c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2}.$$

We do not need any more equations than this since we can readily see that even in this case, we have the same equations we did when computing $C(V_3)$. Hence, $C(V_4) = C(V_3)$ and $\dim(C(V_4)) - \dim(V_4) = 3$. The difference for this case was that we "ran out of room": the (4,0)-entry of the matrix $\partial_1(x)$ is 0. This forced the coefficient on $m_7$ in the expansion of $\partial_1(x)$ to be 0. This essentially destroyed any chance of our centralizer getting larger since entries $x_{4,1} = x_{3,2} = x_{2,3} = x_{1,4}$ of the matrix $x$ depend solely on $m_7$. However, suppose that we were working with larger matrices, say $7 \times 7$, over a larger field $\mathbb{Z}_7$. Then the 5th anti-diagonal would not be the main anti-diagonal and there might be a possibility for growth.

To this end, temporarily let $p = 7$ and consider the pattern

$$\alpha_3 = \begin{pmatrix} \bullet & \bullet & \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and write $V_0 = V(\alpha_3)$. Further, fix non-zero scalars $c_8, c_9, c_{10} \in \mathbb{Z}_7$ and define the matrices

$$m_4 = \begin{pmatrix} 0 & 0 & 0 & c_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad m_5 = \begin{pmatrix} 0 & 0 & 0 & c_9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$m_6 = \begin{pmatrix} 0 & 0 & 0 & c_{10} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $V_3 = \langle V_0, m_4, m_5, m_6 \rangle$. Using the same reasoning as before, we define

$$v_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_8 c_{10} & 0 & 0 \\ 0 & 0 & 0 & c_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_8 + c_9 c_{10} & 0 & 0 \\ 0 & 0 & 0 & c_9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{and} \quad v_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_9 + c_{10}^2 & 0 & 0 \\ 0 & 0 & 0 & c_{10} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and note that $C(V_3) = \langle V(\beta_3), v_2, v_3, v_4 \rangle$ and $\dim(C(V_3)) - \dim(V_3) = 4$. We now have three cases which mirror the previous three that we investigated.

Case 5.1 ($p = 7$): Fix $c_{11} \in \mathbb{Z}_7$ and define

$$m_7 = \begin{pmatrix} 0 & 0 & 0 & c_{11} & c_9 + c_{10}^2 & 0 & 0 \\ 0 & 0 & 0 & c_{10} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $V_4 = \langle V_0, m_4, m_5, m_6, m_7 \rangle$. We have $C(V_4) = C(V_3)$ for the same reasons $C(V_4) = C(V_3)$ in the case when $p = 5$.

Case 5.2 ($p = 7$): Fix $c_{11}, c_{12} \in \mathbb{Z}_5$ and define

$$m_7 = \begin{pmatrix} 0 & 0 & 0 & c_{12} & c_8 + c_9c_{10} + c_{11}(c_9 + c_{10}^2) & 0 & 0 \\ 0 & 0 & 0 & c_9 + c_{10}c_{11} & 0 & 0 & 0 \\ 0 & 0 & c_{11} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $V_4 = \langle V_0, m_4, m_5, m_6, m_7 \rangle$. We have $C(V_4) = C(V_3)$ for the same reasons $C(V_4) = C(V_3)$ in the case when $p = 5$.

Case 5.3 ($p = 7$): Fix $c_{11}, c_{12}, c_{13} \in \mathbb{Z}_5$ and define

$$m_7 = \begin{pmatrix} 0 & 0 & 0 & c_{13} & c_8c_{10} + c_{11}(c_8 + c_9c_{10}) + c_{12}(c_9 + c_{10}^2) & 0 & 0 \\ 0 & 0 & 0 & c_8 + c_9c_{11} + c_{10}c_{12} & 0 & 0 & 0 \\ 0 & 0 & c_{12} & 0 & 0 & 0 & 0 \\ 0 & c_{11} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $V_4 = \langle V, m_4, m_5, m_6, m_7 \rangle$.

Let us now compute the centralizer of $V_4$. By Lemma 14 in Chapter II, each element of $C(V_4)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & x_{0,5} & 0 \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 & 0 & 0 \\ x_{5,0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_4)$ if and only if $\partial_1(x) \in V_4$ and $\partial_2(x) \in V_4$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 & 0 \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 & 0 & 0 \\ x_{5,0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{and} \qquad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & x_{0,5} & 0 & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 & 0 & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_4$ if and only if

$$x_{1,3} = c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2} + c_{13} \cdot x_{5,0}$$

$$x_{4,1} = c_{11} \cdot x_{5,0}$$

$$x_{3,2} = c_{12} \cdot x_{5,0}$$

$$x_{2,3} = (c_8 + c_9 c_{11} + c_{10} c_{12}) \cdot x_{5,0}$$

$$x_{1,4} = (c_8 c_{10} + c_{11}(c_8 + c_9 c_{10}) + c_{12}(c_9 + c_{10}^2))x_{5,0},$$

and that $\partial_2(x) \in V_4$ if and only if

$$x_{0,4} = c_8 \cdot x_{3,1} + c_9 \cdot x_{2,2} + c_{10} \cdot x_{1,3} + c_{13} \cdot x_{4,1}$$

$$x_{3,2} = c_{11} \cdot x_{4,1}$$

$$x_{2,3} = c_{12} \cdot x_{4,1}$$

$$x_{1,4} = (c_8 + c_9 c_{11} + c_{10} c_{12}) \cdot x_{4,1}$$

$$x_{0,5} = (c_8 c_{10} + c_{11}(c_8 + c_9 c_{10}) + c_{12}(c_9 + c_{10}^2))x_{4,1}.$$

Hence, $x \in C(V_4)$ if and only if

$$x_{1,3} = c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2} + c_{13} \cdot x_{5,0}$$

$$x_{0,4} = c_8 \cdot x_{3,1} + c_9 \cdot x_{2,2} + c_{10} \cdot x_{1,3} + c_{13} \cdot x_{4,1}$$

$$= c_8 c_{10} \cdot x_{4,0} + (c_8 + c_9 c_{10}) x_{3,1} + (c_9 + c_{10}^2) x_{2,2} + c_{11} c_{13} \cdot x_{5,0}$$

$$x_{4,1} = c_{11} \cdot x_{5,0}$$

$$x_{3,2} = c_{12} \cdot x_{5,0} = c_{11} \cdot x_{4,1} = c_{11}^2 \cdot x_{5,0}$$

$$x_{2,3} = (c_8 + c_9 c_{11} + c_{10} c_{12}) \cdot x_{5,0} = c_{12} \cdot x_{4,1} = c_{11} c_{12} \cdot x_{5,0}$$

$$x_{1,4} = (c_8 c_{10} + c_{11}(c_8 + c_9 c_{10}) + c_{12}(c_9 + c_{10}^2)) x_{5,0}$$

$$= (c_8 + c_9 c_{11} + c_{10} c_{12}) \cdot x_{4,1} = c_{11}(c_8 + c_9 c_{11} + c_{10} c_{12}) x_{5,0}$$

$$x_{0,5} = (c_8 c_{10} + c_{11}(c_8 + c_9 c_{10}) + c_{12}(c_9 + c_{10}^2)) x_{4,1}$$

$$= c_{11}(c_8 c_{10} + c_{11}(c_8 + c_9 c_{10}) + c_{12}(c_9 + c_{10}^2)) x_{5,0}.$$

Notice that the above equations imply the following equations

$$0 = (c_{12} - c_{11}^2) x_{5,0}$$

$$0 = ((c_8 + c_9 c_{11} + c_{10} c_{12}) - c_{11} c_{12}) x_{5,0}$$

$$0 = ((c_8 c_{10} + c_{11}(c_8 + c_9 c_{10}) + c_{12}(c_9 + c_{10}^2)) - c_{11}(c_8 + c_9 c_{11} + c_{10} c_{12})) x_{5,0}.$$

The above equations force us to consider the following coefficient relationships

$$c_{12} = c_{11}^2$$

$$(c_8 + c_9 c_{11} + c_{10} c_{12}) = c_{11} c_{12}$$

$$(c_8 c_{10} + c_{11}(c_8 + c_9 c_{10}) + c_{12}(c_9 + c_{10}^2)) = c_{11}(c_8 + c_9 c_{11} + c_{10} c_{12}).$$

If one of the above coefficient relationships does not hold, this would imply that $x_{5,0} = 0$. This would yield that $C(V_4) = C(V_3)$ and $\dim(C(V_4)) - \dim(V_4) = 3$ as in the other cases.

Now suppose that all three coefficient relationships hold. In other words, suppose that the following equations hold:

$$c_{12} = c_{11}^2$$
$$(c_8 + c_9 c_{11} + c_{10} c_{12}) = c_{11} c_{12} = c_{11}^3$$
$$(c_8 c_{10} + c_{11}(c_8 + c_9 c_{10}) + c_{12}(c_9 + c_{10}^2)) = c_{11}(c_8 + c_9 c_{11} + c_{10} c_{12}) = c_{11}^4.$$

This further yields

$$x_{1,3} = c_8 \cdot x_{4,0} + c_9 \cdot x_{3,1} + c_{10} \cdot x_{2,2} + c_{13} \cdot x_{5,0}$$
$$x_{0,4} = c_8 c_{10} \cdot x_{4,0} + (c_8 + c_9 c_{10}) x_{3,1} + (c_9 + c_{10}^2) x_{2,2} + c_{11} c_{13} \cdot x_{5,0}$$
$$x_{4,1} = c_{11} \cdot x_{5,0}$$
$$x_{3,2} = c_{11}^2 \cdot x_{5,0}$$
$$x_{2,3} = c_{11} c_{12} \cdot x_{5,0} = c_{11}^3 \cdot x_{5,0}$$
$$x_{1,4} = c_{11}(c_8 + c_9 c_{11} + c_{10} c_{12}) x_{5,0} = c_{11}^4 \cdot x_{5,0}$$
$$x_{0,5} = c_{11}(c_8 c_{10} + c_{11}(c_8 + c_9 c_{10}) + c_{12}(c_9 + c_{10}^2)) x_{5,0} = c_{11}^5 \cdot x_{5,0}.$$

Thus, we define

$$v_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_{11}c_{13} & c_{11}^5 & 0 \\ 0 & 0 & 0 & c_{13} & c_{11}^4 & 0 & 0 \\ 0 & 0 & 0 & c_{11}^3 & 0 & 0 & 0 \\ 0 & 0 & c_{11}^2 & 0 & 0 & 0 & 0 \\ 0 & c_{11} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and note that $C(V_4) = \langle V(\beta_3), v_1, v_2, v_3, v_4 \rangle$. We now make an observation. Suppose that $p$ and $q$ are primes such that $p < q$. Suppose further that $V_i$ is a doubly invariant subspace of $M_{p \times p}(\mathbb{Z}_p)$ and $V_i'$ is a doubly invariant subspace of $M_{q \times q}(\mathbb{Z}_q)$ such that $V_i$ and $V_i'$ are "of the same form" (as in Cases 5.3 ($p = 5$) and 5.3 ($p = 7$)). Finally, assume that $V_i$ contains a matrix which has a non-zero entry in the $(p-1)$th row or the $(p-1)$th column. Then it may very well be the case that $C(V_{i+1}) = C(V_i)$ while $C(V_{i+1}') \neq C(V_i')$. Stating this more loosely, whether or not one is "running into the side of the matrix" affects the computation of the centralizer of a given subspace.

## 6.4 The Fourth Wave Pattern

This pattern will not be fully explored. After one iteration of our algorithm, we will define Case 1, Case 2, Case 3, and Case 4 since there will be four different ways to define the matrix $m_1$. After one iteration of our algorithm for Cases 1, 2, 3, and 4, we will be able to treat them simultaneously. Hence, we will combine them into a single case, which we will call Case 5. Case 5 will then be subdivided into Case 5.1, Case 5.2, Case 5.3, Case 5.4, Case 5.5, and Case 5.6 since there will be six different ways to define the matrices $m_2$ and $m_3$. We shall iterate our algorithm once for Case 5.1 through Case 5.6. Case 6 will be derived from a subset of the subcases of Case 5. Case 6 shall be subdivided into Case 6.1, Case 6.2, Case 6.3, and Case 6.4 due to the different ways one will be able to define the matrices $m_4$, $m_5$, and $m_6$. We will iterate our algorithm once on each of these cases. Similarly, Case 7 will be derived from a subset of the subcases of Case 6. We shall iterate our algorithm once for Case 7 and then cease investigation of this pattern.

We define the pattern

$$\alpha_4 = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and write $V_0 = V(\alpha_4)$. Clearly, $C(V_0) = V(\beta_4)$ where

$$\beta_4 = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 \\ \bullet & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we want the one-dimensional good subspaces of $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$. From our construction of such spaces in Chapter III, we know that these take four different forms. Hence, we must break up our investigation into cases.

Case 1: Fix scalars $c_1, c_2, c_3, c_4 \in \mathbb{Z}_5$ such that $(c_1, c_2, c_3, c_4) \neq (0, 0, 0, 0)$ and define

$$m_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_4 \\ 0 & 0 & 0 & c_3 & 0 \\ 0 & 0 & c_2 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_1 = \langle V_0, m_1 \rangle$.

Let us now compute the centralizer of $V_1$. By Lemma 14 in Chapter II, each element of $C(V_1)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_1)$ if and only if $\partial_1(x) \in V_1$ and $\partial_2(x) \in V_1$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_1$ if and only if

$$x_{4,1} = 0$$

$$x_{3,2} = 0$$

$$x_{2,3} = 0$$

$$x_{1,4} = 0.$$

At this point, we do not need any more equations to conclude that $C(V_1) = C(V_0)$ and $\dim(C(V_1)) - \dim(V_1) = 4$. Note once again that "running into the wall of the matrix" has obstructed the centralizer from getting larger. In fact, we might have even had $c_2 = c_1^2$, $c_3 = c_1^3$, and $c_4 = c_1^4$ and it would still be the case that $C(V_1) = C(V_0)$. In other words, even if the entries on the anti-diagonal of $m_1$ had formed a geometric sequence it would still be the case that $C(V_1) = C(V_0)$.

Case 2: Fix scalars $c_1, c_2, c_3 \in \mathbb{Z}_5$ such that $(c_1, c_2, c_3) \neq (0, 0, 0)$ and define

$$m_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_3 \\ 0 & 0 & 0 & c_2 & 0 \\ 0 & 0 & c_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_1 = \langle V_0, m_1 \rangle$.

Let us now compute the centralizer of $V_1$. By Lemma 14 in Chapter II, each element of $C(V_1)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_1)$ if and only if $\partial_1(x) \in V_1$ and $\partial_2(x) \in V_1$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_1$ if and only if

$$x_{3,2} = c_1 \cdot x_{4,1}$$

$$x_{2,3} = c_2 \cdot x_{4,1}$$

$$x_{1,4} = c_3 \cdot x_{4,1},$$

and that $\partial_2(x) \in V_1$ if and only if

$$x_{4,1} = 0$$

$$x_{2,3} = c_1 \cdot x_{3,2}$$

$$x_{1,4} = c_2 \cdot x_{3,2}$$

$$0 = c_3 \cdot x_{3,2}.$$

Hence, $x \in C(V_1)$ if and only if

$$x_{4,1} = 0$$

$$x_{3,2} = c_1 \cdot x_{4,1} = 0$$

$$x_{2,3} = c_2 \cdot x_{4,1} = 0$$

$$x_{1,4} = c_3 \cdot x_{4,1} = 0.$$

Thus, $C(V_1) = C(V_0)$ and $\dim(C(V_1)) - \dim(V_1) = 4$.

Case 3: Fix scalars $c_1, c_2 \in \mathbb{Z}_5$ such that $(c_1, c_2) \neq (0,0)$ and define

$$m_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & c_1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_1 = \langle V_0, m_1 \rangle$.

Let us now compute the centralizer of $V_1$. By Lemma 14 in Chapter II, each element of $C(V_1)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_1)$ if and only if $\partial_1(x) \in V_1$ and $\partial_2(x) \in V_1$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_1$ if and only if

$$x_{4,1} = 0$$

$$x_{2,3} = c_1 \cdot x_{3,2}$$

$$x_{1,4} = c_2 \cdot x_{3,2},$$

and that $\partial_2(x) \in V_1$ if and only if

$$x_{4,1} = 0$$

$$x_{3,2} = 0$$

$$x_{1,4} = c_1 \cdot x_{2,3}$$

$$0 = c_2 \cdot x_{2,3}.$$

Hence, $x \in C(V_1)$ if and only if

$$x_{4,1} = 0$$

$$x_{3,2} = 0$$

$$x_{2,3} = c_1 \cdot x_{3,2} = 0$$

$$x_{1,4} = c_2 \cdot x_{3,2} = 0.$$

Thus, $C(V_1) = C(V_0)$ and $\dim(C(V_1)) - \dim(V_1) = 4$.

Case 4: Fix a non-zero scalar $c_1 \in \mathbb{Z}_5$ and define

$$m_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_1 = \langle V_0, m_1 \rangle$.

Let us now compute the centralizer of $V_1$. By Lemma 14 in Chapter II, each element of $C(V_1)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_1)$ if and only if $\partial_1(x) \in V_1$ and $\partial_2(x) \in V_1$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_1$ if and only if

$$x_{4,1} = 0$$

$$x_{3,2} = 0$$

$$x_{1,4} = c_2 \cdot x_{2,3},$$

and that $\partial_2(x) \in V_1$ if and only if

$$x_{4,1} = 0$$

$$x_{3,2} = 0$$

$$x_{2,3} = 0$$

$$0 = c_1 \cdot x_{1,4}.$$

Hence, $x \in C(V_1)$ if and only if

$$x_{4,1} = 0$$

$$x_{3,2} = 0$$

$$x_{2,3} = 0$$

$$x_{1,4} = c_1 \cdot x_{2,3} = 0.$$

Thus, $C(V_1) = C(V_0)$ and $\dim(C(V_1)) - \dim(V_1) = 4$.

Notice how in each of the above subcases, it does not matter what type of matrix we use to create $V_1$. We always obtained the result that $C(V_1) = C(V_0)$. This is in sharp contrast to other patterns. In cases where there exists a non-zero $c \in \mathbb{Z}_p$ such that $(i$,0)-entry of the matrix we use to create $V_1$ is 1 and the $(i-j, j)$-entry is $c^j$ then it is always the case that $C(V_1) \neq C(V_0)$.

All of the above cases ended in the same conclusion. Hence, we combine them into a single case, Case 5.

Case 5: We now seek the 2-dimensional good subspaces of $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$. From Chapter III, there are six different ways we can do this depending on what our set of leading positions is. That is, our set of leading positions can either be $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, or $\{3,4\}$. As such, we have six subcases to consider.

Case 5.1: Suppose that our set of leading positions is $\{1,2\}$, fix $c_5, c_6, c_7, c_8, c_9, c_{10} \in \mathbb{Z}_5$ such that $(c_5, c_6, c_7) \neq (0,0,0)$ and $(c_8, c_9, c_{10}) \neq (0,0,0)$ and define

$$m_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_7 \\ 0 & 0 & 0 & c_6 & 0 \\ 0 & 0 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad m_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_{10} \\ 0 & 0 & 0 & c_9 & 0 \\ 0 & 0 & c_8 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_2 = \langle V_0, m_2, m_3 \rangle$.

Let us now compute the centralizer of $V_2$. By Lemma 14 in Chapter II, each element of $C(V_2)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_2)$ if and only if $\partial_1(x) \in V_2$ and $\partial_2(x) \in V_2$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_2$ if and only if

$$x_{3,2} = c_8 \cdot x_{4,1}$$

$$x_{2,3} = c_9 \cdot x_{4,1}$$

$$x_{1,4} = c_{10} \cdot x_{4,1},$$

and that $\partial_2(x) \in V_2$ if and only if

$$x_{2,3} = c_5 \cdot x_{4,1} + c_8 \cdot x_{3,2}$$

$$x_{1,4} = c_6 \cdot x_{4,1} + c_9 \cdot x_{3,2}$$

$$0 = c_7 \cdot x_{4,1} + c_{10} \cdot x_{3,2}.$$

Hence, $x \in C(V_2)$ if and only if

$$x_{3,2} = c_8 \cdot x_{4,1}$$

$$x_{2,3} = c_5 \cdot x_{4,1} + c_8 \cdot x_{3,2} = (c_5 + c_8^2)x_{4,1} = c_9 \cdot x_{4,1}$$

$$x_{1,4} = c_6 \cdot x_{4,1} + c_9 \cdot x_{3,2} = (c_6 + c_8c_9)x_{4,1} = c_{10} \cdot x_{4,1}$$

$$0 = c_7 \cdot x_{4,1} + c_{10} \cdot x_{3,2} = (c_7 + c_8c_{10})x_{4,1}.$$

Note that the above system of equations implies the following system of equations

$$0 = ((c_5 + c_8^2) - c_9)x_{4,1}$$

$$0 = ((c_6 + c_8 c_9) - c_{10})x_{4,1}$$

$$0 = (c_7 + c_8 c_{10})x_{4,1}.$$

The above equations force us to consider the following coefficient relationships

$$(c_5 + c_8^2) = c_9$$

$$(c_6 + c_8 c_9) = c_{10}$$

$$-c_7 = c_8 c_{10}.$$

If one of the above coefficient relationships does not hold, then it must be that $x_{4,1} = 0$. This would imply that $C(V_2) = C(V_0)$ and $\dim(C(V_2)) - \dim(V_2) = 3$.

If all three of the above coefficient relationships hold, then we define

$$v_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_6 + c_8 c_9 \\ 0 & 0 & 0 & c_5 + c_8^2 & 0 \\ 0 & 0 & c_8 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

and note that $C(V_2) = \langle C(\beta_4), v_1 \rangle$ and $\dim(C(V_2)) - \dim(V_2) = 4$. Hence, we fix

$c_{11}, c_{12}, c_{13} \in \mathbb{Z}_5$ and define

$$m_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_{13} \\ 0 & 0 & 0 & c_{12} & c_6 + c_8 c_9 \\ 0 & 0 & c_{11} & c_5 + c_8^2 & 0 \\ 0 & 0 & c_8 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

and $V_3 = \langle V_0, m_2, m_3, m_4 \rangle$.

Let us now compute the centralizer of $V_3$. By Lemma 14 in Chapter II, each element of $C(V_3)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,0} & x_{3,1} & x_{3,2} & x_{3,3} & 0 \\ x_{4,0} & x_{4,1} & x_{4,2} & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_3)$ if and only if $\partial_1(x) \in V_3$ and $\partial_2(x) \in V_3$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,0} & x_{3,1} & x_{3,2} & x_{3,3} & 0 \\ x_{4,0} & x_{4,1} & x_{4,2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & 0 \\ x_{3,1} & x_{3,2} & x_{3,3} & 0 & 0 \\ x_{4,1} & x_{4,2} & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_3$ if and only if

$$x_{3,2} = c_8 \cdot x_{4,1}$$

$$x_{2,3} = c_9 \cdot x_{4,1}$$

$$x_{1,4} = c_{10} \cdot x_{4,1}$$

$$x_{4,2} = 0$$

$$x_{3,3} = 0$$

$$x_{2,4} = 0,$$

and that $\partial_2(x) \in V_3$ if and only if

$$x_{2,3} = c_5 \cdot x_{4,1} + c_8 \cdot x_{3,2} + c_{11} \cdot x_{4,2} = c_5 \cdot x_{4,1} + c_8 \cdot x_{3,2}$$

$$x_{1,4} = c_6 \cdot x_{4,1} + c_9 \cdot x_{3,2} + c_{12} \cdot x_{4,2} = c_6 \cdot x_{4,1} + c_9 \cdot x_{3,2}$$

$$x_{3,3} = c_8 \cdot x_{4,2}$$

$$x_{2,4} = (c_5 + c_8^2)x_{4,2}$$

$$0 = c_7 \cdot x_{4,1} + c_{10} \cdot x_{3,2}.$$

Note that it must the case that $x_{4,2} = x_{3,3} = x_{2,4} = 0$. Also, for the other entries, we have the same equations we obtained when we computed $C(V_2)$. Thus, $C(V_3) = C(V_2)$ and $\dim(C(V_3)) - \dim(V_3) = 3$. This is as far as this case has been investigated.

Case 5.2: Suppose that our set of leading positions is $\{1, 3\}$, fix $c_5, c_6, c_7, c_8, c_9 \in \mathbb{Z}_5$ such that $(c_5, c_6, c_7) \neq (0, 0, 0)$ and $(c_8, c_9) \neq (0, 0)$ and define

$$m_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_7 \\ 0 & 0 & 0 & c_6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & c_5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad m_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_9 \\ 0 & 0 & 0 & c_8 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_2 = \langle V_0, m_2, m_3 \rangle$.

Let us now compute the centralizer of $V_2$. By Lemma 14 in Chapter II, each element of $C(V_2)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_2)$ if and only if $\partial_1(x) \in V_2$ and $\partial_2(x) \in V_2$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_2$ if and only if

$$x_{4,1} = 0$$

$$x_{2,3} = c_8 \cdot x_{3,2}$$

$$x_{1,4} = c_9 \cdot x_{3,2},$$

and that $\partial_2(x) \in V_2$ if and only if

$$\begin{aligned} x_{3,2} &= c_5 \cdot x_{4,1} \\ x_{1,4} &= c_6 \cdot x_{4,1} + c_8 \cdot x_{2,3} \\ 0 &= c_7 \cdot x_{4,1} + c_9 \cdot x_{2,3}. \end{aligned}$$

Hence, $x \in C(V_2)$ if and only if

$$x_{4,1} = 0$$

$$x_{3,2} = c_5 \cdot x_{4,1} = 0$$

$$x_{2,3} = c_8 \cdot x_{3,2} = 0$$

$$x_{1,4} = c_9 \cdot x_{3,2} = 0.$$

Thus, $C(V_2) = C(V_0)$ and $\dim(C(V_2)) - \dim(V_2) = 3$. This is as far as this case has been investigated.

Case 5.3: Suppose that our set of leading positions is $\{1, 4\}$. Fix $c_5, c_6, c_7, c_8 \in \mathbb{Z}_5$ such that $(c_5, c_6, c_7) \neq (0, 0, 0)$ and $c_8 \neq 0$ and define

$$m_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_6 & 0 & 0 \\ 0 & c_5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad m_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_8 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_2 = \langle V_0, m_2, m_3 \rangle$.

Let us now compute the centralizer of $V_2$. By Lemma 14 in Chapter II, each element of $C(V_2)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_2)$ if and only if $\partial_1(x) \in V_2$ and $\partial_2(x) \in V_2$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_2$ if and only if

$$x_{4,1} = 0$$

$$x_{3,2} = 0$$

$$x_{1,4} = c_8 \cdot x_{2,3},$$

and that $\partial_2(x) \in V_2$ if and only if

$$x_{3,2} = c_5 \cdot x_{4,1}$$

$$x_{2,3} = c_6 \cdot x_{4,1}$$

$$0 = c_7 \cdot x_{4,1} + c_8 \cdot x_{1,4}.$$

Hence, $x \in C(V_2)$ if and only if

$$x_{4,1} = 0$$

$$x_{3,2} = 0$$

$$x_{2,3} = c_6 \cdot x_{4,1} = 0$$

$$x_{1,4} = c_8 \cdot x_{2,3} = 0.$$

Thus, $C(V_2) = C(V_0)$ and $\dim(C(V_2)) - \dim(V_2) = 3$. This is as far as this case has been investigated.

Case 5.4: Suppose that our set of leading positions is $\{2,3\}$, fix $c_5, c_6, c_7, c_8 \in \mathbb{Z}_5$ such that $(c_5, c_6) \neq (0,0)$ and $(c_7, c_8) \neq (0,0)$ and define

$$m_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_6 \\ 0 & 0 & 0 & c_5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad m_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_8 \\ 0 & 0 & 0 & c_7 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_2 = \langle V_0, m_2, m_3 \rangle$.

Let us now compute the centralizer of $V_2$. By Lemma 14 in Chapter II, each element of $C(V_2)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_2)$ if and only if $\partial_1(x) \in V_2$ and $\partial_2(x) \in V_2$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_2$ if and only if

$$x_{2,3} = c_5 \cdot x_{4,1} + c_7 \cdot x_{3,2}$$

$$x_{1,4} = c_6 \cdot x_{4,1} + c_8 \cdot x_{3,2},$$

and that $\partial_2(x) \in V_2$ if and only if

$$x_{4,1} = 0$$

$$x_{1,4} = c_5 \cdot x_{3,2} + c_7 \cdot x_{2,3}$$

$$0 = c_6 \cdot x_{3,2} + c_8 \cdot x_{2,3}.$$

Hence, $x \in C(V_2)$ if and only if

$$x_{4,1} = 0$$

$$x_{2,3} = c_5 \cdot x_{4,1} + c_7 \cdot x_{3,2} = c_7 \cdot x_{3,2}$$

$$x_{1,4} = c_6 \cdot x_{4,1} + c_8 \cdot x_{3,2} = c_8 \cdot x_{3,2}$$

$$= c_5 \cdot x_{3,2} + c_7 \cdot x_{2,3} = (c_5 + c_7^2)x_{3,2}$$

$$0 = c_6 \cdot x_{3,2} + c_8 \cdot x_{2,3} = (c_6 + c_7 c_8)x_{3,2}.$$

Note that the above system of equations implies the following system of equations

$$0 = ((c_5 + c_7^2) - c_8)x_{3,2}$$

$$0 = (c_6 + c_7 c_8)x_{3,2}.$$

The above equations force us to consider the following coefficient relationships

$$(c_5 + c_7^2) = c_8$$

$$-c_6 = c_7 c_8.$$

If one of the above coefficient relationships does not hold, then it must be that $x_{3,2} = 0$. This would imply that $C(V_2) = C(V_0)$ and $\dim(C(V_2)) - \dim(V_2) = 3$.

If both of the above coefficient relationships hold, then we define

$$v_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_5 + c_7^2 \\ 0 & 0 & 0 & c_7 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

and note that $C(V_2) = \langle C(\beta_4), v_1 \rangle$ and $\dim(C(V_2)) - \dim(V_2) = 4$. This is as far as this case has been investigated.

Case 5.5: Suppose that our set of leading positions is $\{2, 4\}$. Fix $c_5, c_6, c_7 \in \mathbb{Z}_5$ such that $(c_5, c_6) \neq (0, 0)$ and $c_7 \neq 0$ and define

$$m_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad m_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_7 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_2 = \langle V_0, m_2, m_3 \rangle$.

Let us now compute the centralizer of $V_2$. By Lemma 14 in Chapter II, each

element of $C(V_2)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_2)$ if and only if $\partial_1(x) \in V_2$ and $\partial_2(x) \in V_2$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_2$ if and only if

$$\begin{aligned} x_{3,2} &= c_5 \cdot x_{4,1} \\ x_{1,4} &= c_6 \cdot x_{4,1} + c_7 \cdot x_{2,3}, \end{aligned}$$

and that $\partial_2(x) \in V_2$ if and only if

$$\begin{aligned} x_{4,1} &= 0 \\ x_{2,3} &= c_5 \cdot x_{3,2} \\ 0 &= c_6 \cdot x_{3,2} + c_7 \cdot x_{1,4}. \end{aligned}$$

Hence, $x \in C(V_2)$ if and only if

$$x_{4,1} = 0$$

$$x_{3,2} = c_5 \cdot x_{4,1} = 0$$

$$x_{2,3} = c_5 \cdot x_{3,2} = 0$$

$$x_{1,4} = c_6 \cdot x_{4,1} + c_7 \cdot x_{2,3} = 0.$$

Thus, $C(V_2) = C(V_0)$ and $\dim(C(V_2)) - \dim(V_2) = 3$. This is as far as this case has been investigated.

Case 5.6: Suppose that our set of leading positions is $\{3, 4\}$. Fix non-zero scalars $c_5, c_6 \in \mathbb{Z}_5$ and define

$$m_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad m_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $V_2 = \langle V_0, m_2, m_3 \rangle$.

Let us now compute the centralizer of $V_2$. By Lemma 14 in Chapter II, each element of $C(V_2)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_2)$ if and only if $\partial_1(x) \in V_2$ and $\partial_2(x) \in V_2$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_2$ if and only if

$$x_{4,1} = 0$$
$$x_{1,4} = c_5 \cdot x_{3,2} + c_6 \cdot x_{2,3},$$

and that $\partial_2(x) \in V_2$ if and only if

$$x_{3,2} = 0$$
$$0 = c_5 \cdot x_{2,3} + c_6 \cdot x_{1,4}.$$

Hence, $x \in C(V_2)$ if and only if

$$x_{4,1} = 0$$

$$x_{3,2} = 0$$

$$x_{1,4} = c_5 \cdot x_{3,2} + c_6 \cdot x_{2,3} = c_6 \cdot x_{2,3}$$

$$0 = c_5 \cdot x_{2,3} + c_6 \cdot x_{1,4} = (c_5 + c_6^2)x_{2,3}.$$

Hence, we have two possibilities. If $-c_5 \neq c_6^2$, then $x_{2,3} = 0$ and $C(V_2) = C(V_0)$. This further implies that $\dim(C(V_2)) - \dim(V_2) = 3$. If $-c_5 = c_6^2$, then we define

$$v_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and note that $C(V_2) = \langle C(V_0), v_1 \rangle$. This is as far as this case has been investigated.

We now make an observation from our survey of the different centralizers for $V_2$ from Case 5.1 through Case 5.6. When our set of leading positions consists of two integers which differ only by 1 ($\{1,2\}, \{2,3\}, \{3,4\}$), there is always the possibility that $C(V_2) \neq C(V_1)$. However, this possibility always depends on one or more coefficient relationships. Also, when our set of leading positions consists of two integers that differ by more than 1 ($\{1,3\}, \{1,4\}, \{2,4\}$), then it is always the case that $C(V_2) = C(V_1)$.

Case 6: We shall now investigate the 3-dimensional good subspaces of $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$. From Chapter III, there are four different ways we can do this depending on what our set of leading positions is. That is, our set of leading positions can either be $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, or $\{2, 3, 4\}$. As such, we have four subcases to consider.

Case 6.1: Suppose that our set of leading positions is $\{1, 2, 3\}$. Fix $c_1, c_2, c_3, c_4, c_5, c_6 \in \mathbb{Z}_5$ such that $(c_1, c_2) \neq (0, 0)$, $(c_3, c_4) \neq (0, 0)$, and $(c_5, c_6) \neq (0, 0)$ and define

$$m_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_4 \\ 0 & 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$m_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_6 \\ 0 & 0 & 0 & c_5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $V_3 = \langle V_0, m_1, m_2, m_3 \rangle$.

Let us now compute the centralizer of $V_3$. By Lemma 14 in Chapter II, each

element of $C(V_3)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_3)$ if and only if $\partial_1(x) \in V_3$ and $\partial_2(x) \in V_3$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_3$ if and only if

$$x_{2,3} = c_3 \cdot x_{4,1} + c_5 \cdot x_{3,2}$$

$$x_{1,4} = c_4 \cdot x_{4,1} + c_6 \cdot x_{3,2},$$

and that $\partial_2(x) \in V_3$ if and only if

$$x_{1,4} = c_1 \cdot x_{4,1} + c_3 \cdot x_{3,2} + c_5 \cdot x_{2,3}$$

$$0 = c_2 \cdot x_{4,1} + c_4 \cdot x_{3,2} + c_6 \cdot x_{2,3}.$$

Hence, $x \in C(V_3)$ if and only if

$$x_{2,3} = c_3 \cdot x_{4,1} + c_5 \cdot x_{3,2}$$

$$x_{1,4} = c_1 \cdot x_{4,1} + c_3 \cdot x_{3,2} + c_5 \cdot x_{2,3} = (c_1 + c_3c_5)x_{4,1} + (c_3 + c_5^2)x_{3,2}$$

$$= c_4 \cdot x_{4,1} + c_6 \cdot x_{3,2}$$

$$0 = c_2 \cdot x_{4,1} + c_4 \cdot x_{3,2} + c_6 \cdot x_{2,3} = (c_2 + c_3c_6)x_{4,1} + (c_4 + c_5c_6)x_{3,2}.$$

Note that the above system of equations implies the following system of equations

$$0 = (c_1 + c_3c_5 - c_4)x_{4,1} + (c_3 + c_5^2 - c_6)x_{3,2}$$

$$0 = (c_2 + c_3c_6)x_{4,1} + (c_4 + c_5c_6)x_{3,2}.$$

Much like Cases 4.1.1 through 4.1.4, the above identities grant the possibility of many different centralizers of $V_3$ that are dependent upon the various relationships of the coefficients involved. This is as far as this case has been investigated.

Case 6.2: Suppose that our set of leading positions is $\{1, 2, 4\}$. Fix $c_1, c_2, c_3, c_4, c_5 \in \mathbb{Z}_5$ such that $(c_1, c_2) \neq (0, 0)$, $(c_3, c_4) \neq (0, 0)$, and $c_5 \neq 0$ and define

$$m_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$m_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $V_3 = \langle V_0, m_1, m_2, m_3 \rangle$.

Let us now compute the centralizer of $V_3$. By Lemma 14 in Chapter II, each element of $C(V_3)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_3)$ if and only if $\partial_1(x) \in V_3$ and $\partial_2(x) \in V_3$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_3$ if and only if

$$x_{3,2} = c_3 \cdot x_{4,1}$$

$$x_{1,4} = c_4 \cdot x_{4,1} + c_5 \cdot x_{2,3},$$

and that $\partial_2(x) \in V_3$ if and only if

$$x_{2,3} = c_1 \cdot x_{4,1} + c_3 \cdot x_{3,2}$$

$$0 = c_2 \cdot x_{4,1} + c_4 \cdot x_{3,2} + c_5 \cdot x_{1,4}.$$

Hence, $x \in C(V_3)$ if and only if

$$x_{3,2} = c_3 \cdot x_{4,1}$$

$$x_{2,3} = c_1 \cdot x_{4,1} + c_3 \cdot x_{3,2} = (c_1 + c_3^2)x_{4,1}$$

$$x_{1,4} = c_4 \cdot x_{4,1} + c_5 \cdot x_{2,3} = (c_4 + c_5(c_1 + c_3^2))x_{4,1}$$

$$0 = c_2 \cdot x_{4,1} + c_4 \cdot x_{3,2} + c_5 \cdot x_{1,4} = (c_2 + c_3c_4 + c_5(c_4 + c_5(c_1 + c_3^2)))x_{4,1}.$$

Hence, there are two possibilities. First, if $c_2 + c_3c_4 + c_5(c_4 + c_5(c_1 + c_3^2)) \neq 0$, then $x_{4,1} = 0$. This would imply that $C(V_3) = C(V_0)$ and $\dim(C(V_3)) - \dim(V_3) = 2$. Second, if $c_2 + c_3c_4 + c_5(c_4 + c_5(c_1 + c_3^2)) = 0$, then we define

$$v_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_2 + c_3c_4 + c_5(c_4 + c_5(c_1 + c_3^2)) \\ 0 & 0 & 0 & c_1 + c_3^2 & 0 \\ 0 & 0 & c_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and note that $C(V_3) = \langle C(V_0), v_1 \rangle$ and $\dim(C(V_3)) - \dim(V_3) = 3$. This is as far as this case has been investigated.

Case 6.3: Suppose that our set of leading positions is $\{1, 3, 4\}$. Fix $c_1, c_2, c_3, c_4 \in \mathbb{Z}_5$ such that $(c_1, c_2) \neq (0, 0)$, $c_3 \neq 0$, and $c_4 \neq 0$ and define

$$m_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$m_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $V_3 = \langle V_0, m_1, m_2, m_3 \rangle$.

Let us now compute the centralizer of $V_3$. By Lemma 14 in Chapter II, each element of $C(V_3)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_3)$ if and only if $\partial_1(x) \in V_3$ and $\partial_2(x) \in V_3$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_3$ if and only if

$$x_{4,1} = 0$$

$$x_{1,4} = c_3 \cdot x_{3,2} + c_4 \cdot x_{2,3},$$

and that $\partial_2(x) \in V_3$ if and only if

$$x_{3,2} = c_1 \cdot x_{4,1}$$

$$0 = c_2 \cdot x_{4,1} + c_3 \cdot x_{2,3} + c_4 \cdot x_{1,4}.$$

Hence, $x \in C(V_3)$ if and only if

$$x_{4,1} = 0$$

$$x_{3,2} = c_1 \cdot x_{4,1} = 0$$

$$x_{1,4} = c_3 \cdot x_{3,2} + c_4 \cdot x_{2,3} = c_4 \cdot x_{2,3}$$

$$0 = c_2 \cdot x_{4,1} + c_3 \cdot x_{2,3} + c_4 \cdot x_{1,4} = (c_3 + c_4^2)x_{2,3}.$$

Hence, there are two possibilities. First, if $c_3 + c_4^2 \neq 0$, then $x_{4,1} = 0$. This would imply that $C(V_3) = C(V_0)$ and $\dim(C(V_3)) - \dim(V_3) = 2$. Second, if $c_3 + c_4^2 = 0$, then we define

$$v_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and note that $C(V_3) = \langle C(V_0), v_1 \rangle$ and $\dim(C(V_3)) - \dim(V_3) = 3$. This is as far as this case has been investigated.

Case 6.4: Suppose that our set of leading positions is $\{2, 3, 4\}$. Fix $c_1, c_2, c_3, \in \mathbb{Z}_5$ such that $c_1 \neq 0$, $c_2 \neq 0$, and $c_3 \neq 0$ and define

$$m_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$m_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $V_3 = \langle V_0, m_1, m_2, m_3 \rangle$.

Let us now compute the centralizer of $V_3$. By Lemma 14 in Chapter II, each element of $C(V_3)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_3)$ if and only if $\partial_1(x) \in V_3$ and $\partial_2(x) \in V_3$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_3$ if and only if

$$x_{1,4} = c_1 \cdot x_{4,1} + c_2 \cdot x_{3,2} + c_3 \cdot x_{2,3},$$

and that $\partial_2(x) \in V_3$ if and only if

$$\begin{aligned} x_{4,1} &= 0 \\ 0 &= c_1 \cdot x_{3,2} + c_2 \cdot x_{2,3} + c_3 \cdot x_{1,4}. \end{aligned}$$

Hence, $x \in C(V_3)$ if and only if

$$x_{4,1} = 0$$

$$x_{1,4} = c_1 \cdot x_{4,1} + c_2 \cdot x_{3,2} + c_3 \cdot x_{2,3} = c_2 \cdot x_{3,2} + c_3 \cdot x_{2,3}$$

$$0 = c_1 \cdot x_{3,2} + c_2 \cdot x_{2,3} + c_3 \cdot x_{1,4} = (c_1 + c_2 c_3)x_{3,2} + (c_2 + c_3^2)x_{2,3}.$$

The last equation listed above makes this case much like Case 1. It should be clear that the last equation above grants the possibility of different centralizers of $V_3$ dependent upon the various relationships of the coefficients involved. This is as far as this case has been investigated.

Case 7: Finally we shall investigate the 4-dimensional good subspaces of $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. From Chapter III, there is only one such subspace. Fix non-zero scalars $c_1, c_2, c_3, c_4 \in \mathbb{Z}_5$ and define $V_4 = \langle V_0, m_1, m_2, m_3, m_4 \rangle$ where

$$m_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$m_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad m_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & c_4 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us now compute the centralizer of $V_4$. By Lemma 14 in Chapter II, each element of $C(V_4)$ has the form

$$x = \begin{pmatrix} x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} \\ x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \end{pmatrix}.$$

Note that $x \in C(V_4)$ if and only if $\partial_1(x) \in V_4$ and $\partial_2(x) \in V_4$. Observe that

$$\partial_1(x) = \begin{pmatrix} x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & 0 \\ x_{3,0} & x_{3,1} & x_{3,2} & 0 & 0 \\ x_{4,0} & x_{4,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \partial_2(x) = \begin{pmatrix} x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & 0 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & 0 \\ x_{2,1} & x_{2,2} & x_{2,3} & 0 & 0 \\ x_{3,1} & x_{3,2} & 0 & 0 & 0 \\ x_{4,1} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Lemma 7 in Chapter III, we see that $\partial_1(x) \in V_4$ if and only if

$$x_{1,4} = c_2 \cdot x_{4,1} + c_3 \cdot x_{3,2} + c_4 \cdot x_{2,3},$$

and that $\partial_2(x) \in V_4$ if and only if

$$0 = c_1 \cdot x_{4,1} + c_2 \cdot x_{3,2} + c_3 \cdot x_{2,3} + c_4 \cdot x_{1,4}.$$

Hence, $x \in C(V_4)$ if and only if

$$\begin{aligned} x_{1,4} &= c_2 \cdot x_{4,1} + c_3 \cdot x_{3,2} + c_4 \cdot x_{2,3} \\ 0 &= c_1 \cdot x_{4,1} + c_2 \cdot x_{3,2} + c_3 \cdot x_{2,3} + c_4 \cdot x_{1,4} \\ &= (c_1 + c_2 c_4) x_{4,1} + (c_2 + c_3 c_4) x_{3,2} + (c_3 + c_4^2) x_{2,3}. \end{aligned}$$

The last equation listed above makes this case much like Case 4.1.1 through Case 4.1.4. It should be clear that the above equation grants the possibility of different centralizers of $V_4$ dependent upon the various relationships of the coefficients involved. This is as far as this case has been investigated.

CHAPTER VII

# CONCLUDING REMARKS

In this thesis we have described a method to enumerate and compute all of the doubly-invariant subspaces of $\mathcal{M}$. We have done this completely for the cases $p = 2$ and $p = 3$ and partially for the case $p = 5$. While the case $p = 5$ is much more complex than previous cases, it is a much more interesting case. For future investigations, we now describe a few ideas which may prove useful.

First, one should complete the case $p = 5$ as it will grant more insight into the form of the subspaces of $\mathcal{V}$ as well as the centralizers of the those subspaces. Once the case $p = 5$ is completed, one should compare cases $p = 2$, $p = 3$, and $p = 5$. We hope that this examination will inspire some theorems which will apply to the case of an arbitrary prime $p$.

Second, let $V_i \in \mathcal{V}$ and $m \in C(V_i) - V_i$ and write $V_{i+1} = \langle V_i, m \rangle$. Suppose that $C(V_i)$ is a proper subspace of $C(V_{i+1})$. Then we have observed time and again that there is almost always a matrix contained in $V_i$ whose non-zero entries have a geometric relationship. This type of relationship has almost always guaranteed that $C(V_i)$ is a proper subspace of $C(V_{i+1})$. This connection needs to be explored further.

Third, fix primes $p$ and $q$ such that $p < q$. Suppose $V \in M_{p\times p}(\mathbb{Z}_p)$ and $V' \in M_{q\times q}(\mathbb{Z}_q)$ are doubly invariant matrices which "have the same form". Assume further that $V$ contains a matrix which has a non-zero entry in the $(p-1)$th row or $(p-1)$th column. We have observed the centralizer of $V$ might be different than the centralizer of $V'$. The circumstances under which this type of difference occurs needs to be studied as well.

Finally, note that many of the computations and certainly the linear transformations have a very mechanical feel to them. With further refinement, it is quite possible that these may become automated. If a computer were able to do more of the computational work for this investigation, it would greatly aid the investigator. This would free up the investigator to focus on looking for theorems concerning the subspaces of $\mathcal{V}$.

We hope that the results in this thesis and the above comments will prove useful in further investigations of this problem.

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